The Studying of The Generalized Finite Difference on the Uniform Grid

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ABSTRACT

In this work, the accuracy of conventional and generelized finite difference methods are compared. The two methods are applied on the same uniform grid. The first and second-order partial derivatives using 3 nodes at the accuracy of second-order for the conventional finite difference methods seem to have the same value of accuracy to the generalized finite difference methods for the first and second-order derivatives using 9 nodes star. The generalized finite difference coefficients were calculated using Cholesky method.

Introduction

The based of generalized finite difference were published by Jensen (1972) , using fully arbitrary mesh with six point stars. Using Taylor series expansion he obtained the GFD formulae which approached derivatives up to the second-order. Many researchers are then concentrate on developing the GFD methods in order to improve the accuracy and the stability of the GFD methods. Nowaday, the research in the topic of GFD are on going in many application, for example in the exploration seismic, in Benito (2015), the GFD methods are used to simulate the elastic wave propagation with the perfectly matched layer(PML) boundary condition in the heterogeneous media and in Salete (2017), the stability of PML boundary condition is studied.

This report contains my initial research on the generalized finite difference methods. In order to compare the accuracy of the generalized finite difference (GFD) and the conventional finite difference (FD) methods the uniform grids are necessary since the FD is been restricted to the uniform grids. The approximation coefficients of FD and GFD methods are

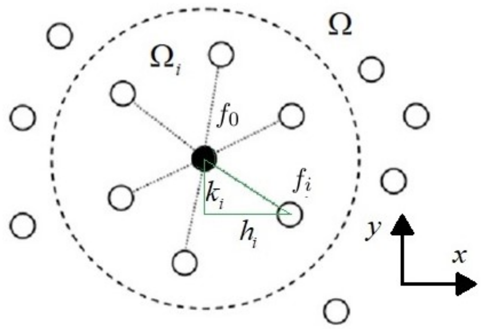

Figure 1: In order to find the approximation scheme in the generalized finite difference, the set of nodes called star is formed which consists of the central node and the neighbor nodes. The neighbor nodes value are used in order to calculate the derivatives of the function at the central nodes similarly to the conventional finite difference.

calculated from Taylor series expansion and Cholesky methods, respectively. The first part of the report is the introduction to the generalized finite difference method. The second part provide the results and discussion on the experiment and the last part is the conclusions.

Generalized Finite Difference

The Taylor series expansion around a point $P(x_0, y_0)$ of the function $f(x, y)$, in a given domain, can be expressed in the form: Benito (2001)

$$f = f_0 + h\frac{\partial f_0}{\partial x} + k\frac{\partial f_0}{\partial y} + \frac{h^2}{2}\frac{\partial^2 f_0}{\partial x^2} + \frac{k^2}{2}\frac{\partial^2 f_0}{\partial y^2} + kh\frac{\partial^2 f_0}{\partial x \partial y} + O(\rho^3) \tag{1}$$

where $f = f(x, y), f_0 = f(x_0, y_0), h = x - x_0, k = y - y_0$ and $\rho = \sqrt{h^2 + k^2}$.

If in Eq.1 the terms over the second order are ignored, it is possible to define the function B as:

$$B = \sum_{i=1}^{N} \left[\left\{ f_0 - f_i + h_i\frac{\partial f_0}{\partial x} + k_i\frac{\partial f_0}{\partial y} + \frac{h_i^2}{2}\frac{\partial^2 f_0}{\partial x^2} + \frac{k_i^2}{2}\frac{\partial^2 f_0}{\partial y^2} + k_i h_i\frac{\partial^2 f_0}{\partial x \partial y} \right\} w_i \right]^2 \tag{2}$$

where w_i is a positive weighting function, $f_i = f(x_i, y_i), h_i = x_i - x_0,\ k_i = y_i - y_0$, and N is the number of neighbor nodes in the star. Figure 1 shows the configuration of star.

Minimizing function B with respect to $\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}, \frac{\partial^2 f_0}{\partial x^2}, \frac{\partial^2 f_0}{\partial y^2}$, and $\frac{\partial^2 f_0}{\partial x \partial y}$.

$$\frac{\partial B}{\partial \mathbf{D}} = 0 \tag{3}$$

$$\mathbf{D} = \left\{ \frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}, \frac{\partial^2 f_0}{\partial x^2}, \frac{\partial^2 f_0}{\partial y^2}, \frac{\partial^2 f_0}{\partial x \partial y} \right\}^T$$

The set of five equations which is the result of minimizing B respect to $\mathbf{D}$ make up the linear system equations. For example, The first equation of the system is as following:

$$\begin{aligned} \frac{\partial B}{\partial \{\partial f_0/\partial x\}} = f_0 \sum_{i=1}^N w_i^2 h_i - \sum_{i=1}^N f_i w_i^2 h_i + \frac{\partial f_0}{\partial x} \sum_{i=1}^N w_i^2 h_i^2 + \frac{\partial f_0}{\partial y} \sum_{i=1}^N w_i h_i k_i \\ + \frac{\partial^2 f_0}{\partial x^2} \sum_{i=1}^N w_i^2 \frac{h_i^3}{2} + \frac{\partial^2 f_0}{\partial y^2} \sum_{i=1}^N w_i^2 \frac{k_i^2 h_i}{2} + \frac{\partial f_0}{\partial x \partial y} \sum_{i=1}^N w_i^2 h_i^2 k_i = 0 \end{aligned} \tag{4}$$

Note that, it is possible to include more terms in the Taylor series expansion up to higher order of accuracy but the system of equations will be larger and the increasing of accuracy is not compansate to the addition computational cost.

Using all five equations to form the system of equations, we have:

$$\mathbf{AD} = \mathbf{b} \tag{5}$$

$$\mathbf{A} = \begin{bmatrix} \sum_{i=1}^N h_i^2 w_i^2 & \sum_{i=1}^N h_i k_i w_i^2 & \sum_{i=1}^N \frac{h_i^3}{2} w_i^2 & \sum_{i=1}^N \frac{h_i k_i^2}{2} w_i^2 & \sum_{i=1}^N h_i^2 k_i^2 w_i^2 \\ & \sum_{i=1}^N k_i^2 w_i^2 & \sum_{i=1}^N \frac{h_i^2 k_i}{2} w_i^2 & \sum_{i=1}^N \frac{k_i^3}{2} w_i^2 & \sum_{i=1}^N h_i k_i^2 w_i \\ & & \sum_{i=1}^N \frac{h_i^4}{4} w_i^2 & \sum_{i=1}^N \frac{h_i^2 k_i^2}{4} w_i^2 & \sum_{i=1}^N \frac{h_i^3 k_i}{2} w_i^2 \\ & SYM & & \sum_{i=1}^N \frac{k_i^4}{4} w_i^2 & \sum_{i=1}^N \frac{h_i k_i^3}{2} w_i^2 \\ & & & & \sum_{i=1}^N h_i^2 k_i^2 w_i^2 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} \sum_{i=1}^N (-f_0 + f_i) h_i^2 w_i^2 \\ \sum_{i=1}^N (-f_0 + f_i) k_i^2 w_i^2 \\ \sum_{i=1}^N (-f_0 + f_i) \frac{h_i^2}{2} w_i^2 \\ \sum_{i=1}^N (-f_0 + f_i) \frac{k_i^2}{2} w_i^2 \\ \sum_{i=1}^N (-f_0 + f_i) h_i k_i w_i^2 \end{bmatrix}$$

The coefficients matrix $\mathbf{A}$ is calculated by the space coordinates of the each neighbor node. On the other hand, the vector $\mathbf{b}$ is related to the physical quantities and and space coordinates of nodes in the star.

By the fact that the matrix of coefficients $\mathbf{A}$ is symmetric positive definite (proof in Appendix A.), it is then be able to use the Cholesky method to solve the system of equations. The aim is to obtain the decomposition in upper and lower triangular matrices:

$$\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathbf{T}} \tag{6}$$

The coefficients matrix $\mathbf{L}$ is denoted by $l(i,j)$ with $i,j = 1,...,5$. The system may be imployed as follows:

$$\mathbf{L}(\mathbf{L}^{\mathbf{T}}\mathbf{D}) = \mathbf{b} \tag{7}$$

The first is solving for the value of $\mathbf{L}^{\mathbf{T}}\mathbf{D}$ and denotes the value as $\mathbf{Y}$.

$$\mathbf{L}\mathbf{Y} = \mathbf{b} \tag{8}$$

Once the vector $\mathbf{Y}$ has been established, then it is possible to solve the followig equation for the vector $\mathbf{D}$:

$$\mathbf{L}^{\mathbf{T}}\mathbf{D} = \mathbf{Y} \tag{9}$$

It is possible to obtain the following explicit difference formulae: Benito (2007)

$$\mathbf{D}(k) = \frac{1}{l(k,k)}\left[\mathbf{Y}(k) - \sum_{i=1}^{5-k} l(k+i,k)\mathbf{D}(k+i)\right] \qquad (k = 1,...,5) \tag{10}$$

$$\mathbf{Y}(k) = \frac{1}{l(k,k)}\left[-f_0\sum_{i=1}^{5} M(k,i)c_i + \sum_{j=1}^{N} f_j\left(\sum_{i=1}^{5} M(k,i)d_{ij}\right)\right] \qquad (k = 1,...,5)$$

where

$$M(i,j) = (-1)^{i+j}\frac{1}{l(i,i)}\sum_{k=1}^{i-1} l(i,k)M(k,j) \qquad \text{for } j < i, \ \ i,j = 1,...,5$$

$$M(i,j) = \frac{1}{l(i,i)} \qquad \text{for } j = i, \ \ i,j = 1,...,5$$

$$M(i,j) = 0 \qquad \text{for } j > i, \ \ i,j = 1,...,5$$

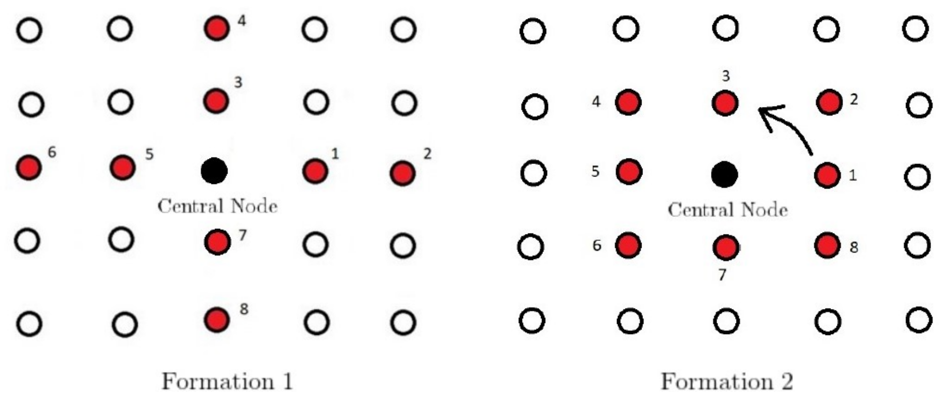

Figure 2: The star of GFD consists for 8 neighbor nodes and a central node. The indices of nodes are increased in counter-clockwise direction in the star formation 1.

$$c_i = \sum_{j=1}^{N} d_{ij}$$

$$d_{j1} = h_j w_j^2 \qquad d_{j2} = k_j w_j^2 \qquad d_{j3} = \frac{h_j^2}{2} w_j^2 \qquad d_{j4} = \frac{k_j^2}{2} w_j^2 \qquad d_{j5} = h_j k_j w_j^2$$

In order to express the unknown partial differences in the explicit form using the above formulae, the explicit form of star equation is obtained:

$$\mathbf{D}(k) = -m_{0k} f_0 + \sum_{i=1}^{N} m_{ik} f_i \qquad (k = 1, ..., 5) \tag{11}$$

where m_{0k} and m_{ik} are the GFD coefficient at the central node and the neighbor node i^{th} with respect to the unknown $\mathbf{D}(k)$, respectively.

The method of calculation for GFD coefficients in form of matrices is shown in Appendix A. This generalized finite difference scheme has the second order of accuracy the proof is contained in Appendix B which corresponded well to the results in the next section.

Results and Discussion

Using the weighting function $w(h_i, k_i) = \frac{1}{\sqrt{h_i^2 + k_i^2}^3}$ and the 2 formations of 9 nodes star as shown in figure 1. The experiment was carried out on the uniform grid with $nx = nz = 51$ and $dx = dz = 0.1$. The function $f(x, y) = sin(x/6)cos(y/6)$, as shown in figure 2, is the

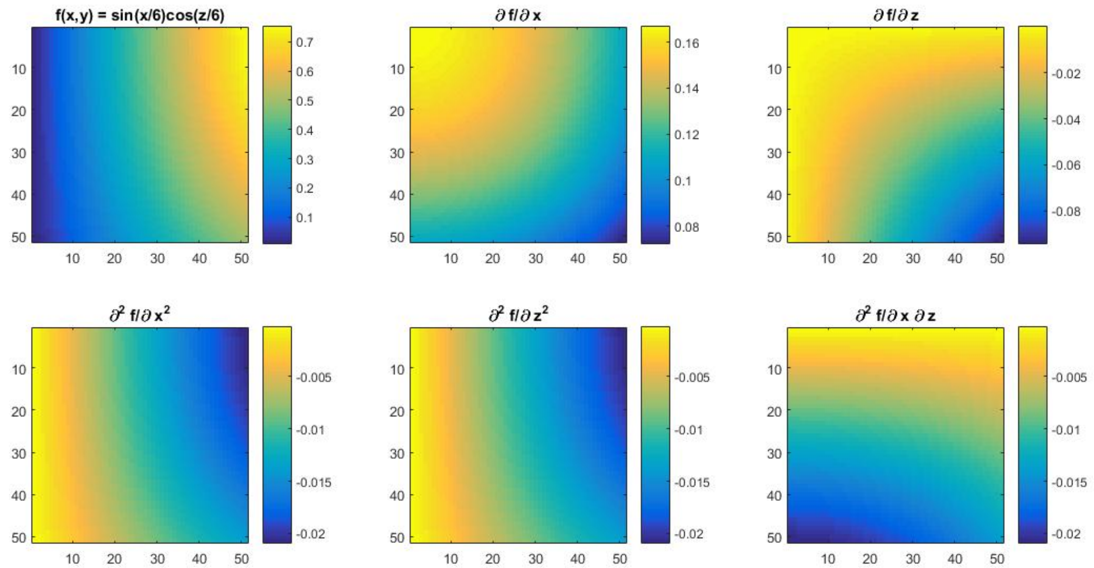

Figure 3: The exact solution and its derivatives.

test function whose derivatives are the exact solutions in the experiment.

the errors in the experiment are the relative errors between the numerical results and the exact solution.

$$\text{Relative Error\%} = \frac{\sqrt{\sum_{i=1}^{N}(f_i^n - f_i^e)^2}}{\sqrt{\sum_{i=1}^{N}(f_i^e)^2}} \times 100 \tag{12}$$

where f_i^n and f_i^e denote numerical and exact solutions at the i^{th} test point, respectively, which can be $\frac{\partial f_0}{\partial x}$, $\frac{\partial f_0}{\partial y}$, $\frac{\partial^2 f_0}{\partial x^2}$, $\frac{\partial^2 f_0}{\partial y^2}$, or $\frac{\partial^2 f_0}{\partial x \partial y}$. N is the total number of nodes in the domain.

The accuracy of GFD 9 nodes star can be compared to the results of 2-order of accuracy in the FD methods as shown in the figures 4 and 5. The FD methods can give the better results using the same amount of nodes than the GFD methods.

The GFD coefficients can be calculated by Cholesky method. The results of using GFD coefficients to calculate the values of $\partial^2 f/\partial z^2$ and $\partial^2 f/\partial y^2$ are corresponded very well to the results that calculated from the direct inverse of $\mathbf{A}$ in equation 5. The FD and GFD coefficients are shown in figures 4 and 5.

Partial Derivatives	GFD Relative Error%	FD Relative Error%	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
$\frac{\partial f_0}{\partial x}$	0.0054	0.0046 (3points) 4.36e-13 (9points)	0	2.392	0.074	0	0	-2.392	-0.074	0	0
$\frac{\partial f_0}{\partial z}$	0.0054	0.0046 (3points) 1.26e-12 (9points)	0	0	0	2.392	0.0747	0	0	-2.392	-0.074
$\frac{\partial^2 f_0}{\partial x^2}$	0.0037	0.0023 (3points) 1.52e-10 (9points)	86.409	40.663	2.541	0	0	40.663	2.541	0	0
$\frac{\partial^2 f_0}{\partial z^2}$	0.0037	0.0023 (3points) 1.09e-10 (9points)	86.409	0	0	40.66	2.5415	0	0	40.663	2.541

Figure 4: The result using the star formation 1.

Partial Derivatives	GFD Relative Error%	FD Relative Error%	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
$\frac{\partial f_0}{\partial x}$	0.0074	0.0046 (3points) 4.36e-13 (9points)	0	2.033	0.254	0	-0.254	-2.033	-0.254	0	0.254
$\frac{\partial f_0}{\partial z}$	0.0074	0.0046 (3points) 1.26e-12 (9points)	0	0	0.254	2.033	0.254	0	-0.254	-2.0332	-0.254
$\frac{\partial^2 f_0}{\partial x^2}$	0.0046	0.0023 (3points) 1.52e-10 (9points)	84.715	42.357	4.235	-8.471	4.235	42.357	4.235	-8.4716	4.235
$\frac{\partial^2 f_0}{\partial z^2}$	0.0046	0.0023 (3points) 1.09e-10 (9points)	84.715	-8.471	4.235	42.357	4.235	-8.471	4.235	42.357	4.235
$\frac{\partial^2 f_0}{\partial x \partial z}$	0.0093	0.0093 (5points)	0	0	12.707	0	-12.707	0	12.707	0	-12.707

Figure 5: The result using the star formation 2.

Conclusion

The 9 nodes star generalized finite difference methods generate the results in the 2-order of accuracy. For the uniform grids, we can say that 9 nodes star GFD methods are similarly to the 3 nodes 1-order derivatives or the 3 node 2-order derivatives in the FD methods in term of the accuracy. The coefficients matrix of the GFD methods are symmetric positive definite (SPD) and can be solved by Cholesky method.

Appendix A

According to Gavete (2017), if (x_0, y_0) are the coordinates of central node of a star and $(x_i, y_i)(i = 1, ..., s)$ are the coordinates of the neighbor nodes of the star and are different, we calculate (h_i, k_i), with $h_i = x_i - x_0$ and $k_i = y_i - y_0$. Matrix A can be written as:

$$\mathbf{A} = \mathbf{PWP^T} \tag{A.1}$$

where

$$\mathbf{P} = \begin{bmatrix} h_1 & h_2 & ..... & h_N \\ k_1 & k_2 & ..... & h_N \\ \frac{h_1^2}{2} & \frac{h_2^2}{2} & ..... & \frac{h_N^2}{2} \\ \frac{k_1^2}{2} & \frac{k_2^2}{2} & ..... & \frac{k_N^2}{2} \\ k_1 h_1 & k_2 h_2 & ..... & k_N h_N \end{bmatrix} \tag{A.2}$$

by including the basis $\{h_i, k_i, \frac{h_i^2}{2}, \frac{k_i^2}{2}, h_i k_i\}$ as column of a matrix $\mathbf{P}$

$$\mathbf{W} = \begin{bmatrix} w_1^2 & & & \\ & w_2^2 & & \\ & & ... & \\ & & & w_N^2 \end{bmatrix}$$

In matrix $\mathbf{P}$ the row vectors are linearly independent. In Matrix $\mathbf{W}$, $w_i^2 > 0$.

Then, matrix $\mathbf{A} = \mathbf{PWP^T}$ is positive definite and it has a unique Cholesky decomposition. The solution of system Eq.5 is unique and the solutions obtained for the derivatives are a linear combination of the function values obtained at the nodes. Then,

$$\mathbf{D} = \mathbf{A^{-1}b} = \mathbf{A^{-1}PW}(\mathbf{f} - f_0\mathbf{1}) \tag{A.3}$$

where $\mathbf{1} = \{1, 1, ..., 1\}^T$ and $\mathbf{f} = \{f_1, f_2, ..., f_N\}^T$

$$\mathbf{D} = \underbrace{\mathbf{A^{-1}PWe_1}}_{\mathbf{m_1}} f_1 + ... + \underbrace{\mathbf{A^{-1}PWe_N}}_{\mathbf{m_N}} f_N - \underbrace{\mathbf{A^{-1}PW}(\mathbf{e_1} + ... + \mathbf{e_N})}_{\mathbf{m_0}} f_0 \tag{A.4}$$

$$\mathbf{D} = -\mathbf{m_0} f_0 + \sum_{i=1}^{N} \mathbf{m_i} f_i \tag{A.5}$$

with

$$\mathbf{m_0} = \sum_{i=1}^{N} \mathbf{m_i} \tag{A.6}$$

where $\mathbf{e_i}(i = 1, ..., N)$ are the vector of the canonical basis, $\mathbf{m_0}$ is the GFD coefficient of the central node, and $\mathbf{m_i}$ is the coefficient of the neghbor node i^{th} in the star.

Appendix B

According to Gavete (2017), Approximation order of the interested GFDM can be shown as following:

$$\mathbf{A^{-1}PW}(\mathbf{f} - f_0\mathbf{1}) = \mathbf{A^{-1}PW}\left(\begin{bmatrix} f_0 + h_1\frac{\partial f_0}{\partial x} + ... + \frac{1}{2}h_1^2\frac{\partial^2 f_0}{\partial x^2} + ... \\ f_0 + h_2\frac{\partial f_0}{\partial x} + ... + \frac{1}{2}h_2^2\frac{\partial^2 f_0}{\partial x^2} + ... \\ . \\ . \\ f_0 + h_N\frac{\partial f_0}{\partial x} + ... + \frac{1}{2}h_N^2\frac{\partial^2 f_0}{\partial x^2} + ... \end{bmatrix} - \begin{bmatrix} f_0 \\ f_0 \\ . \\ . \\ f_0 \end{bmatrix}\right) \tag{B.1}$$

$$= \mathbf{A^{-1}PW} \begin{bmatrix} h_1 \frac{\partial f_0}{\partial x} + ... + \frac{1}{2} h_1^2 \frac{\partial^2 f_0}{\partial x^2} + ... \\ h_2 \frac{\partial f_0}{\partial x} + ... + \frac{1}{2} h_2^2 \frac{\partial^2 f_0}{\partial x^2} + ... \\ . \\ . \\ h_N \frac{\partial f_0}{\partial x} + ... + \frac{1}{2} h_N^2 \frac{\partial^2 f_0}{\partial x^2} + ... \end{bmatrix} - \mathbf{A^{-1}PW} \begin{bmatrix} \Theta(h_1^2, k_2^2) \\ \Theta(h_2^2, k_2^2) \\ . \\ . \\ \Theta(h_N^2, k_N^2) \end{bmatrix} \tag{B.2}$$

by replacing **P** and **W** in Eq. B.2

$$\mathbf{A^{-1}AD} + \Theta(h_i^2, k_i^2) = \mathbf{D} + \Theta(h_i^2, k_i^2) \tag{B.3}$$

Thus, the approximation is of the second order.

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