Nonstandard finite difference for solving wave equation

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ABSTRACT

Nonstandard finite difference (NSFD) are numerical methods that can used to solve differential equations. These methods were implemented in order to achieve higher accuracy compared to conventional finite difference. Nonstandard finite difference scheme is specific to a problem and has no trivial instruction for general problems. Normally it was devised by using information of known-solution of the PDE. When applies to the wave equation, it can provide better results for specific frequency of the wave. In 2D wave equation, the results are better for only plane wave form.

INTRODUCTION

Differential equations can be solved numerically by turning differential equation to difference equation which is discrete equation. A simple method is conventional finite difference. Solutions from this method are approximation. Simplest way to get higher accuracy of solution is decreasing step size. But decreasing step size lead to a higher computational cost.

Another way to achieve higher accuracy without decreasing step size is to modify the usual finite difference using known-solution form. This method can provide better approximate solution or may be exact solution for that solution form. These method, modified form usual finite difference, were called "Nonstandard Finite Difference (NSFD)". Because this method use information from known-solution which is specific for a problem, the NSFD scheme is also specific to that problem.

The application of nonstandard finite difference on wave equation have been done by James B. Cole (Mickens, 2000).

This report contains only theoretical part without numerical result.

MATHEMATICAL MODEL

One Dimensional wave equation

$$\left(\partial_{tt} - v(x)^2\frac{\Delta t^2}{h^2}\partial_{xx}\right)\psi(x,t) = 0 \tag{1}$$

One solution to 1D wave equation 1 is

$$\psi_0(x,t) = e^{i(kx-\omega t)} \tag{2}$$

Standard finite difference or usual finite difference defined by

$$d_x^2 f(x) = \frac{\tilde{d}_x^2 f(x)}{h^2} \tag{3}$$

where h is step size and

$$\tilde{d}_x^2 f(x) = f(x+h) + f(x-h) - 2f(x) \tag{4}$$

which is central finite difference second order.

Now we want to modify this scheme to get a better result for 1D wave equation that its solution is eq. 2. Substitute solution 2 into usual finite difference scheme 3, we get

$$\left(\tilde{d}_t^2 - v(x)^2\frac{\Delta t^2}{h^2}\tilde{d}_x^2\right)\psi_0(x,t) = \epsilon \tag{5}$$

where ϵ is an error from approximation.

To make an error to be zero $\epsilon = 0$, we use the constrain equation

$$\epsilon = 2\psi_0(x,t)\left((\cos(\omega\Delta t) - 1) - v^2(x)\frac{\Delta t^2}{h^2}(\cos(kh) - 1)\right) \tag{6}$$

$$v^2(x)\frac{\Delta t^2}{h^2} = \frac{cos(\omega\Delta t) - 1}{\cos(kh) - 1} \tag{7}$$

Using this constrain (eq. 7) to the scheme 3. We get

$$\left(\tilde{d}_t^2 - \frac{cos(\omega\Delta t) - 1}{\cos(kh) - 1}\tilde{d}_x^2\right)\psi(x,t) = 0 \tag{8}$$

This is nonstandard finite difference scheme on 1D wave equation which provide exact solution $\psi_0 = e^{i(kx-\omega t)}$.

With dispersive relation $v = \omega/k$, equation 8 can be rearranged to

$$\left(\frac{\tilde{d}_t^2}{4cos^2(\omega\Delta t/2)/\omega^2} - v^2(x)\frac{\tilde{d}_x^2}{4\sin^2(k(x)h/2)/k^2}\right)\psi(x,t) = 0 \tag{9}$$

As we can see, it just turn h from usual finite difference (eq. 3) to function of h, k, ω

Two Dimensional wave equation

$$\left(\partial_{tt} - \vec{v}^2\frac{\Delta t^2}{h^2}\nabla^2\right)\psi(x,t) = 0 \tag{10}$$

Turning equation 10 to a finite difference equation

$$(\tilde{D}_t^2 - \vec{v}^2\frac{\Delta t^2}{h^2}\tilde{D}_1^2)\psi(x,t) = 0 \tag{11}$$

where D_1 is defined as

$$\tilde{D}_1^2 = \tilde{d}_x^2 + \tilde{d}_y^2 \tag{12}$$

One known solution is plane wave $\psi_0(x,t) = \Sigma_{k=|k|} e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ Put this solution into equation 11, we get an error equation similar to 1D case.

$$\epsilon = 2e^{i\vec{k}\cdot\vec{x}}(\Sigma(\sin^2(k_i h/2))/h^2 - k^2) \tag{13}$$

But this error depends on the direction of wave vector. We can make the error zero for only one direction. The wave can move in any direction, so this approach does not work.

Another way is to modify the difference operator. Let us use more grid points on space by define

$$\begin{aligned} 2\tilde{D}_2^2 f(x,y) =& f(x+h,y+h) + f(x-h,y+h) \\ &+ f(x+h,y-h) + f(x-h,y-h) - 4f(x,y) \end{aligned}$$

and using

$$\tilde{D}_0^2 = \gamma \tilde{D}_1^2 + (1-\gamma)\tilde{D}_2^2 \tag{14}$$

Assuming that function of h in finite difference depends on only the magnitude of k like in 1D wave scheme.

$$\frac{\tilde{D}_0^2}{4\sin^2(kh/2)} e^{i\vec{k}\cdot\vec{x}} = e^{i\vec{k}\cdot\vec{x}} \tag{15}$$

γ can define as a function of k, h and an angle of wave vector. But after analysis, it seems that the dependence of wave direction on γ is weak. That means it can approximate function $\gamma(k,h)$. Now our scheme dose not depend on the direction anymore.

Two 2D scheme is similar to 1D except that the difference operator was changed to D_0

DISCUSSION

For this mathematical model, solution of 1D NSFD wave equation is exact if the source have only one frequency and source should be a sinusoidal. In 2D wave equation, NSFD give a good approximation for a plane wave at specific frequency.

Future work is to do this in simulation and try to use difference scheme and difference known-solution.

REFERENCES

Mickens, R. E., 2000, Application of nonstandard finite difference scheme: World Scientific, 109–154.