Simulation of Rayleigh-Taylor Instability in Two Dimensions using Finite Difference Method in MATLAB

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ABSTRACT

The interface between two layers of fluids with different densities can become unstable with the heavier fluid layer is on top of the lighter fluid. This phenomenon is called the Rayleigh- Taylor instability. The heavier fluid is pulled down by gravitational force while the lighter fluid pushes up the heavier fluid. Consequently, fluid advection occur. In this study, the time evolution of interface between two fluids is simulated by solving the Navier-Stokes equations using the finite difference approximation.

INTRODUCTION

One of the classic example of hydrodynamic instability is the mixing of two fluids that takes place if a heavy initially lies above a lighter one in a gravitational field. This phenomenon is called the Rayleigh-Taylor instability. The heavier fluid is pulled down by gravitational force while the lighter fluid pushes up the heavier fluid. Consequently, fluid advection occur.

In this project, we interest the evolution of interface between two fluids. The instability that occurred by the difference of densities between two fluids cause the shape of interface advection pattern. The shape of spike and bubble will occur in beginning before amplitude of perturbation will grow up quickly and going to turbulent. That is a reason why we rarely see Rayleigh-Taylor instability in from of spike and bubble in real life. So numerical simulation is the one way to study this phenomena easier.

THEORY OR METHODS

Governing Equations

The conservation of mass equation for incompressible flow is

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

The momentum equation, when the surface tension is neglected, the dynamics viscosity of each fluids are the same, and the one of body force from gravity. Which is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{g} + \mu_0 \nabla^2 \mathbf{u} \tag{2}$$

where $\mathbf{u}$ is partial velocity of fluids, p is pressure, μ_0 is dynamics viscosity, and $\mathbf{g}$ is gravitational force.

Finite Difference Approximation

Derivative term can approximated by using finite difference approximation, they are central finite difference for the first derivative of spatial domain

$$\frac{\partial f}{\partial x} \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \tag{3}$$

central finite difference for the second derivative of spatial domain

$$\frac{\partial^2 f}{\partial x^2} \approx \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{2\Delta x^2} \tag{4}$$

and forward finite difference for the first derivative of time domain

$$\frac{\partial f}{\partial t} \approx \frac{f(t+\Delta t) - f(t)}{\Delta x} \tag{5}$$

Staggered Grid

Staggered grid is rectangular domain which each cell compose by any variables at different location. Middle of cell stored by pressure and density, vertical edges stored by velocity of fluid in x direction, and horizontal edges stored by velocity of fluid in y direction. Shown in Figure 2.1

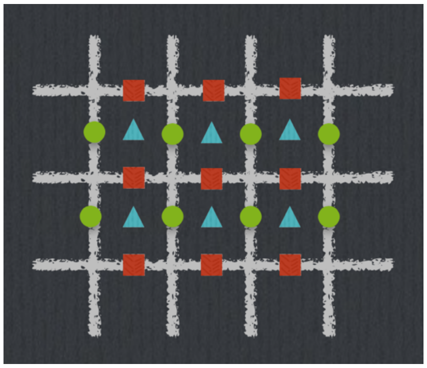

Figure 1: Staggered Grid, triangle is p, pressure, square rectangle is v, the velocity in y direction, and circle is u, the velocity in x direction.

Method of Solving

First, we split the momentum equation(2.2) and calculate the velocity without the pressure term.

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -\mathbf{A}^n + \mathbf{g} + \frac{1}{\rho^n}\mathbf{D}^n \tag{6}$$

and then adding the pressure

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\frac{\nabla_h p}{\rho^n} \tag{7}$$

the superscript n denotes a variable at current time, t, and $n+1$ denotes a variable at next time step, $t + \Delta t$. $\mathbf{A}$ is the discrete approximation of advection term. $\mathbf{D}$ is the discrete approximation of diffusion term. $\mathbf{u}^*$ is temporary velocity. ∇_h is a discrete approximation of gradient.

By taking the divergence of equation (2.7) and using equation (2.1) in discrete approximation at the end of time step that is equation (2.8)

$$\nabla_h \cdot \mathbf{u}^{n+1} = 0 \tag{8}$$

we get the Poisson's equation

$$\nabla_h^2 p = \frac{\rho^n}{\Delta t}\nabla_h \cdot \mathbf{u}^* \tag{9}$$

then we solve Poisson's equation of pressure term and find $\mathbf{u}^{n+1}$. Finally, we advect the density field by using continuity equation add the term of diffusion, which is

$$\frac{\partial \rho}{\partial t} = -(\nabla \cdot \mathbf{u})\rho + \mu_0 \nabla^2 \rho \tag{10}$$

Computational Domain

Computational domain is rectangular domain. L_x, the domain height. L_y, the domain width. Inside of domain composed by any cells which size is $\Delta x \Delta y$, multiple of spatial grid spacing in x and y direction. n_x, the number of rows of domain width. n_y, the number of domain height. In addition, we need to add one row of ghost cell outside the domain for help with implementing boundary conditions. So we get the pressure arrays is dimensioned $p(n_x + 2, n_y + 2)$. Similarly, we need ghost point for tangential velocity. The velocity arrays dimensioned $u(n_x + 1, n_y + 2)$ and $v(n_x + 2, n_y + 1)$.

Boundary Conditions

At boundary of domain, we don't have value of velocity. But we can calculate the tangent velocity on the wall by interpolation of the velocity inside the domain and the ghost velocity, which is given by equation (2.11)

$$u_{wall} = \frac{1}{2}(u_{inside} + u_{ghost}) \tag{11}$$

where u_{wall} is the tangent velocity on the wall, u_{inside} is the velocity inside the domain, and u_{ghost} is the ghost velocity.

In this simulation, we use full slip boundary condition which is the tangent velocity on the wall equal to the velocity inside the domain. In the other hand, we set the ghost velocity equal to the velocity inside the domain.

RESULTS

Perturbed Interface

We study evolution of interface by adding different initial perturbation at the interface between two fluids.By using domain grid size is 32x96, spatial grid spacing in x and y direction are 0.03125, time step is 0.00125, domain height is 3.0, domain width is 1.0, and dynamics viscosity is 0.01.We set the density of heavy fluid is 2.0 and light fluid is 1.0

The interface is advected by follwing the shape of initial perturbation. From Figure 3.1 and Figure 3.2, the interface are cosine shape which have different spatial wavenumber and glowth up of perturbation in time. we can see the glowth up rate is depend on spatial wavenumber.

Unperturbed Interface

We study evolution of interface by no perturbed interface. Similarly, The parameters are the same of perturbed interface case, but the density of heavy fluid is 8.0 and light fluid is 2.0.

In part of unperturbed interface, we expect that interface not change, because Rayleigh-Taylor Instability is unstable fixed point. If no perturbation the interface shouldn't be change. But from Figure 3.4, we can see which have the changing of interface. In this case, we can explain that interface was perturbed by numerical error which occur in each calculation steps.

CONCLUSION

1. Finite difference approximation can be used to simulate Rayleigh-Taylor instability.

2. Shape of perturbation affect the evolution of interface.

3. Rayleigh-Taylor instability can still occur even though there is no interface perturbation. This could be the effect of numerical error.

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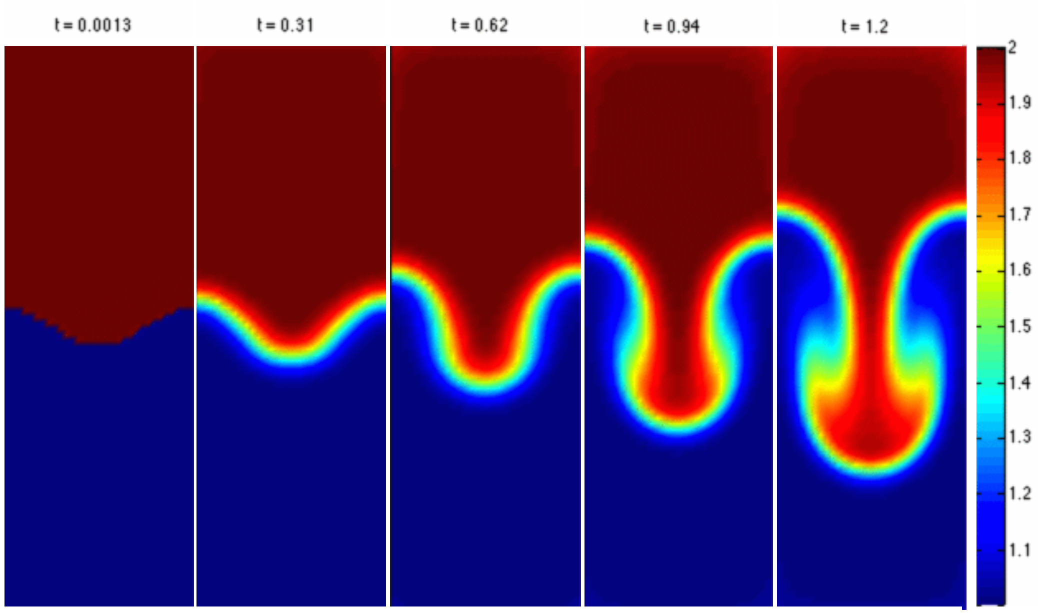

Figure 2: Perturbation is (0.1 x domain width)cos(2πx/domain width), times in sec, and density was shown in different color.

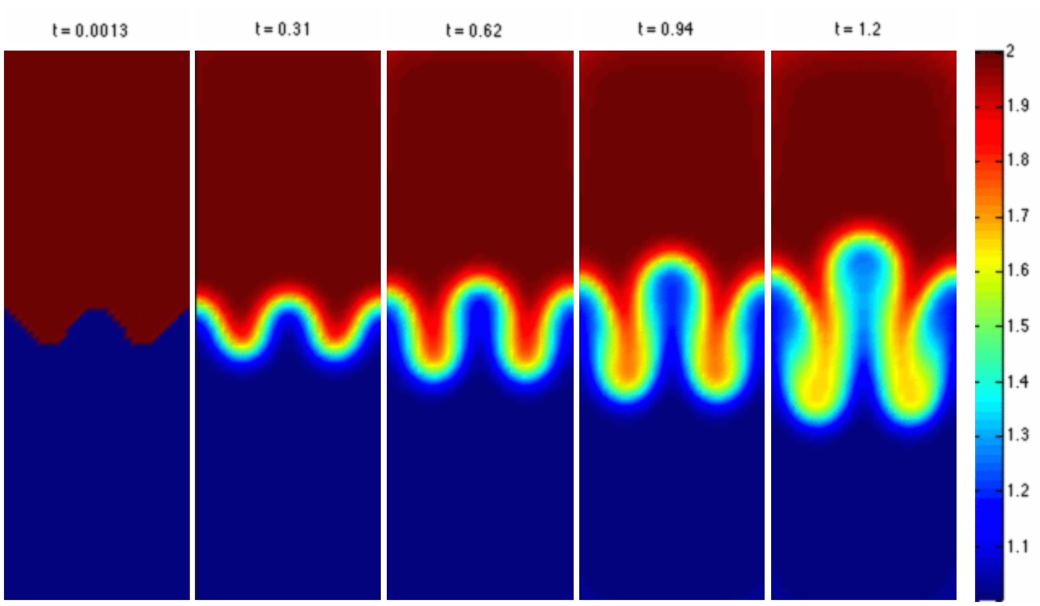

Figure 3: Perturbation is (0.1 x domain width)cos(4πx/domain width), times in sec, and density was shown in different color.

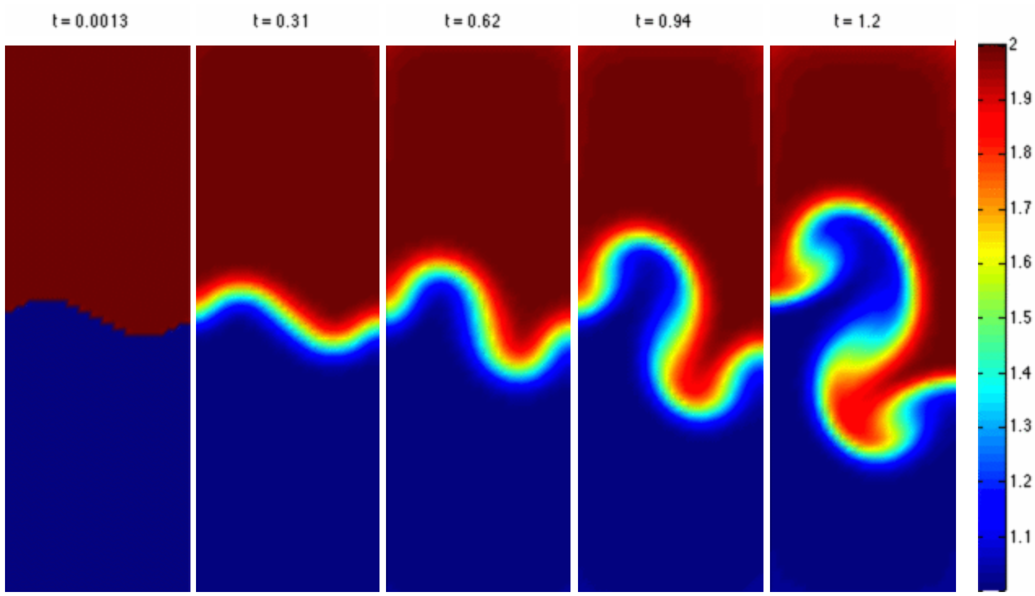

Figure 4: Perturbation is (0.1 x domain width)sin(2πx/domain width), times in sec, and density was shown in different color.

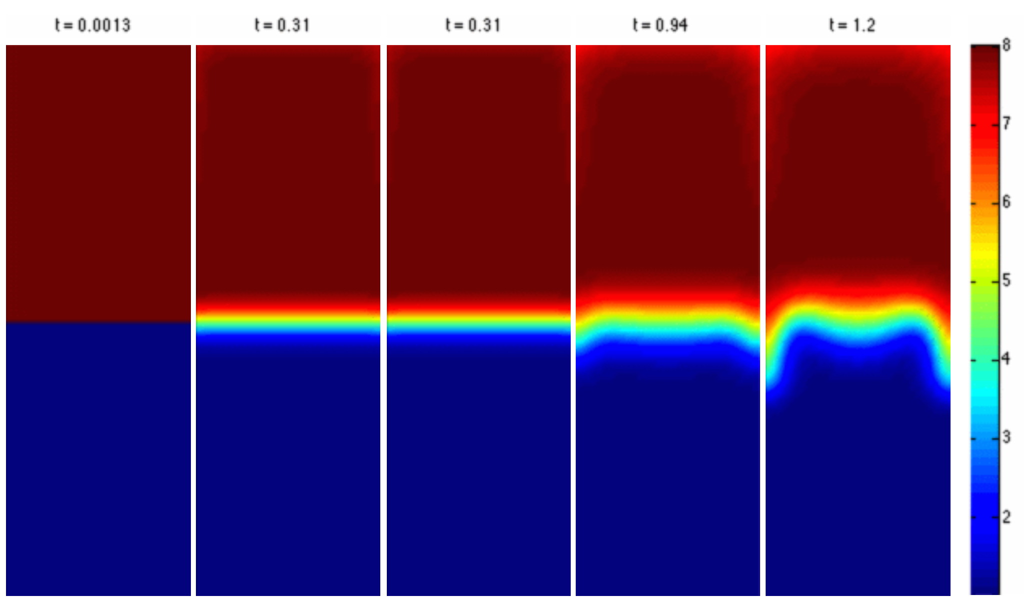

Figure 5: Interface doesn't perturbed, times in sec, and density was shown in different colour.