Morphological Image Operations

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Mathematical Morphology

- “Morphology is the study of how things are put together.” (vocabulary.com)
- “In biology, morphology deals with the form and structure of animals and plants.”
- In image processing, we use mathematical morphology as a tool for extracting image components representing shape such as boundaries or skeletons.
- Mathematical morphology is based on the set theory.
- “Sets in mathematical morphology represent objects in an image.”
- “For example, a binary image can be represented as sets whose components are in the 2D integer space $\mathbb{Z}^2$.”

Gonzalez and Woods (2018, p. 636)
**Mathematical Morphology**

- "Morphological operations are defined in terms of sets."
- "We use mathematical morphology with two types of sets of pixels: objects and structuring elements."
- "A structuring element is a shape, used to probe or interact with a given image, with the purpose of drawing conclusions on how this shape fits or misses the shapes in the image."
- "The choice of a structuring element for a morphological operation influences the information one can obtain about an object in an image."
- The two main characteristics of the structuring element are shape and size.

Gonzalez and Woods (2018, p. 636)
Mathematical Morphology

- Images are rectangular arrays but sets are of arbitrary shape.
- “Applications of morphology in image processing require that sets be embedded in rectangular arrays.”

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Gonzalez and Woods (2018, p. 637)
Basics of Set Theory

- **Union** of sets $A$ and $B$ is denoted by $A \cup B$.
- **Intersection** of set $A$ and $B$ is denoted by $A \cap B$.
- Sets $A$ and $B$ are **disjoint** or mutually exclusive if they have no common elements: $A \cap B = \emptyset$.
- The **complement** of set $A$ is the set of elements not contained in $A$: $A^c = \{w \mid w \not\in A\}$.
- The **difference** of sets $A$ and $B$, denoted $A - B$, is defined as $A - B = \{w \mid w \in A, w \not\in B\} = A \cap B^c$.
- The **reflection** of set $B$, denoted $\hat{B}$, is defined as $\hat{B} = \{w \mid w = -b, \text{ for } b \in B\}$.
- The **translation** of set $A$ by point $z = (z_1, z_2)$, denoted $(A)_z$, is defined as $(A)_z = \{c \mid c = a + z, \text{ for } a \in A\}$.

Gonzalez and Woods (2018, p. 637)
Reflection is rotation by 180 deg about the origin (dot). x’s are “don’t care” elements whose value does not matter.

Gonzalez and Woods (2018, p. 637)
A binary image $I$ contains an object $A$.

A 3x3 structuring element $B$ is used as follows.

From a new image, of the same size as $I$, consisting only the background values (0’s) initially.

Translate $B$ over image $I$. If $B$ is completely contained in $A$, mark the origin of $B$ as foreground (1) in the new image; otherwise, leave it as background (0).

Gonzalez and Woods (2018, p. 638)
Erosion and Dilation

- Erosion and dilation are fundamental to morphological image processing since many morphological operations are based on these two primitive operations.
- Let $A$ and $B$ be sets in $\mathbb{Z}^2$.
- The **erosion** of $A$ by $B$ is defined as

$$A \ominus B = \{ z \mid (B)_z \subseteq A \}$$

where $A$ is the set of foreground pixels, $B$ is a structuring element, and $z$ is the foreground value (1).
- “In words, the erosion of $A$ by $B$ is the set of all points $z$ such that $B$, translated by $z$, is contained in $A$.”
Erosion

Gonzalez and Woods (2018, p. 640)
Erosion

(a) 486x486 binary image with foreground pixels in white.
(b)-(d) Images eroded using square structuring elements of size 11x11, 15x15, 45x45 whose values are all 1.

Gonzalez and Woods (2018, p. 641)
Dilation

- The dilation of $A$ by $B$ is defined as

$$A \oplus B = \left\{ z \mid (\hat{B})_z \cap A \neq \emptyset \right\}$$

- “This equation is based on reflecting $B$ about its origin and translate the reflection by $z$, as in erosion.”
- “The dilation of $A$ by $B$ is the set of all displacements $z$ such that the foreground elements of $B$ overlap at least one element of $A$.”
- “Dilation grows or thickens objects in a binary image.”

Gonzalez and Woods (2018, p. 642)
Dilation

Gonzalez and Woods (2018, p. 640)
Dilation can be used to repair broken characters in an image.

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.

Gonzalez and Woods (2018, p. 644)
Opening and Closing

- “Dilation expands the components of a set and erosion shrinks it.”
- “Opening smoothes the contour of an object, break narrow isthmuses, and eliminates thin protrusions.”
- “Closing also smoothes sections of contours, but fuses narrow break and long thin gulfs, eliminates some holes, and fills gaps in the contours.”
- “The opening of set \( A \) by structuring element \( B \) is defined as”
  \[
  A \circ B = (A \ominus B) \oplus B
  \]
- The closing of set \( A \) by structuring element \( B \) is defined as”
  \[
  A \bullet B = (A \oplus B) \ominus B
  \]

FIGURE 9.8
(a) Image $I$, composed of set object $A$ and background.
(b) Structuring element, $B$.
(c) Translations of $B$ while being contained in $A$. ($A$ is shown dark for clarity.)
(d) Opening of $A$ by $B$.

Gonzalez and Woods (2018, p. 645)
Opening

Gonzalez and Woods (2018, p. 647)
Closing

Gonzalez and Woods (2018, p. 647)
Example: Removing Noise

FIGURE 9.11
(a) Noisy image.
(b) Structuring element.
(c) Eroded image.
(d) Dilation of the erosion (opening of $A$).
(e) Dilation of the opening.
(f) Closing of the opening.
(Original image courtesy of the National Institute of Standards and Technology.)

Gonzalez and Woods (2018, p. 647)
“Hit-or-miss transform (HMT) is a basic tool for shape detection.”

“HMT utilizes two structuring elements: $B_1$ for detecting shapes in the foreground and $B_2$ for detecting shapes in the background.”

The HMT of image $I$ is defined as

$$I \ast B_{1,2} = \left\{ z \mid (B_1)_z \subseteq A \text{ and } (B_2)_z \subseteq A^c \right\}$$

$$= (A \ominus B_1) \cap (A^c \ominus B_2)$$

which is the set of translations $z$ of $B_1$ and $B_2$ such that $B_1$ is contained in $A$ and $B_2$ is contained in $A^c$.

FIGURE 9.12
(a) Image consisting of a foreground (1’s) equal to the union, $A$, of set of objects, and a background of 0’s.
(b) Image with its foreground defined as $A^c$.
(c) Structuring elements designed to detect object $D$.
(d) Erosion of $A$ by $B_1$.
(e) Erosion of $A^c$ by $B_2$.
(f) Intersection of (d) and (e), showing the location of the
Hit-or-Miss Transform

(E) $A^c \ominus B_2$

(F) Origin of $D$

Background

Image: $I \ast B_{1,2} = A \ominus B_1 \cap A^c \ominus B_2$

Gonzalez and Woods (2018, p. 650)
In the previous example, to detect $D$, we have to use two structuring elements.

We can use only one structuring element to detect $D$.

The structuring element is identical to $D$ but having a border of background elements with a width of one pixel.

The HMT using only one structuring element $B$ is then defined as

$$I \odot B = \left\{ z \mid (B)_z \subseteq I \right\}$$

Gonzalez and Woods (2018, p. 651)
HMT using only one SE

Gonzalez and Woods (2018, p. 652)
HMT using only one SE

Gonzalez and Woods (2018, p. 652)
Boundary Extraction

- “The boundary of a set $A$ of foreground pixels, denoted by $\beta(A)$, can be obtained”

$$\beta(A) = A - (A \ominus B)$$

Gonzalez and Woods (2018, p. 653)
- “A hole is defined as a background region surrounded by a connected border of foreground pixels.”
- “Let $A$ denote a set whose elements are 8-connected boundaries, with each boundary enclosing a background region (a hole).”
- “Given a point in each hole, we want to fill all the holes with foreground elements (1’s).”
- “We begin by forming an array $X_0$ of 0’s with the same size as $I$, except at locations of pixels that are known to be holes, which we set to 1.” Then, apply the procedure

$$X_k = \left( X_{k-1} \oplus B \right) \cap I^c \quad k = 1, 2, 3, \ldots$$

where $B$ is a symmetric structuring element

Gonzalez and Woods (2018, p. 653)
“The algorithm terminates if $X_k = X_{k-1}$. Then $X_k$ contains all the filled holes. The set union of $X_k$ and $I$ contains all the filled holes and their boundaries.”

Gonzalez and Woods (2018, p. 654)
Filling Holes

(a) Light microscope of red blood cells. (b) Thresholded image showing outer edges of the blood cells with holes. (c) not (b). (d) Removing the background. (e) (b) or (d). (f) Removing small features, separating touching features, removing edge-touching features. (g) Outlines of the blood cells superimposed on the original image.

Russ and Neal (2016, p. 447)
(a) SEM image of spherical latex particles. (b) Edges enhanced with a gradient operator. (c) Thresholding the edges. (d) Filling the holes. (e) Applying an opening operation to remove small debris.
“Let $A$ be a set of foreground pixels consisting of one or more connected components.”

“Form an image $X_0$ whose elements are 0’s, except at a point in $A$, which we set to 1.”

When $B$ is SE, apply the iterative procedure

$$X_k = (X_{k-1} \oplus B) \cap I \quad k = 1, 2, 3, \ldots$$

The procedure terminates when $X_k = X_{k-1}$.

This equation and the hole filling equation use conditional dilation to limit the growth of set dilation.
Extraction of Connected Components

Gonzalez and Woods (2018, p. 656)
“A set $S$ of points in the Euclidean plane is said to be convex if and only if a straight line segment joining any two points in $S$ lies entirely within $S$.”

“The convex hull $H$ of $S$ is the smallest convex set containing $S$.”

“The convex deficiency of $S$ is defined as the set difference $H - S$.”

“The following morphological algorithm can be used to obtain an approximation of the convex hull of a set $A$ of foreground pixels, embedded in a binary image, $I$.”
Convex Hull

- Let $B^i, i = 1,2,3,4$ be four structuring elements.
- Apply the iterative procedure

$$X_k^i = \left(X_{k-1}^i \ast B^i\right) \cup X_{k-1}^i \quad i = 1, 2, 3, 4 \quad \text{and} \quad k = 1, 2, 3, \ldots$$

with $X_0^i = I$.
- When the procedure converges ($X_0^i = I$), we set $D^i = X_0^i$
- The convex hull of $A$ is the union of the four results:

$$C(A) = \bigcup_{i=1}^{4} D^i$$

Gonzalez and Woods (2018, p. 658)
Convex Hull

Gonzalez and Woods (2018, p. 659)
Thinning

- Thinning of a set $A$ of foreground pixels by a structuring element $B$ is defined in terms of HMT as
  \[
  A \otimes B = A - (A \star B)
  \]
  \[= A \cap (A \star B)^c
  \]

- Given a sequence of SEs: $\{B\} = \{B^1, B^2, \ldots, B^n\}$
- Thinning of $A$ by $\{B\}$ is defined as
  \[
  A \otimes \{B\} = \left(\left(\ldots\left((A \otimes B^1) \otimes B^2\right)\ldots\right) \otimes B^n\right)
  \]

Gonzalez and Woods (2018, p. 660)
Thinning

Gonzalez and Woods (2018, p. 661)
Thinning

$A_6 = A_5 \otimes B^6$

$A_7 = A_6 \otimes B^7 (A_8 = A_7 \otimes B^8 = A_7)$

$A_9 = A_8 \otimes B^1$

$A_{12} = A_{11} \otimes B^4$

$A_{14} = A_{13} \otimes B^6$

$A_{14}$ converted to $m$-connectivity.

Gonzalez and Woods (2018, p. 661)
Thickening is defined as

\[ A \odot B = A \cup (A \ast B) \]

Thickening of \( A \) by \( \{B\} \) is defined as

\[ A \odot \{B\} = \left( \left( A \odot B^1 \right) \odot B^2 \right) \ldots \odot B^n \]

Gonzalez and Woods (2018, p. 661)
A skeleton of set $A$ is denoted by $S(A)$. If $z$ is a point of $S(A)$, a maximum disk $(D)_z$ is the largest disk centered at $z$ and contained in $A$. A maximum disk touches the boundary of $A$ at two or more different places.

Gonzalez and Woods (2018, p. 662)
“The skeleton of set $A$ can be expressed in terms of erosions and openings as follows.”

$$S(A) = \bigcup_{k=0}^{K} S_k(A)$$

where

$$S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B$$

$(A \ominus kB)$ indicates $k$ successive erosions of $A$ by the structuring element $B$.

“$K$ is the last iterative step before $A$ erodes to an empty set, i.e.,”

$$K = \max\left\{ k \mid (A \ominus kB) \neq \emptyset \right\}$$

Gonzalez and Woods (2018, p. 662)
Morphological Reconstruction

- “Morphological reconstruction involves two images and a structuring element.”
- The first image is a **marker** denoted by F. It contains the starting points for reconstruction.
- The second image is a **mask** denoted by G. It is used to constrain the reconstruction.
- “The structuring element is used to define connectivity.”

Gonzalez and Woods (2018, p. 667)
• Assume that the marker $F$ and the mask $G$ are binary images and that $F \subseteq G$.

• “The geodesic dilation of size 1 of the marker image with respect to the mask is defined as”

$$D_G^{(1)} (F) = (F \oplus B) \cap G$$

• “The geodesic dilation of size $n$ of $F$ with respect to $G$ is defined as

$$D_G^{(n)} (F) = D_G^{(1)} (D_G^{(n-1)} (F))$$

where $n \geq 1$ is an integer and

$$D_G^{(0)} (F) = F$$

• “The intersection with the mask $G$ is performed at each step and will limit the growth of the marker $F$.”

Gonzalez and Woods (2018, p. 667)
In this simple example, continuing geodesic dilation will eventually result in an image identical to $G$.

Gonzalez and Woods (2018, p. 668)
“The geodesic erosion of size 1 of marker $F$ with respect to mask $G$ is defined as”

$$E_G^{(1)}(F) = (F \ominus B) \cup G$$

“The geodesic erosion of size $n$ of $F$ with respect to $G$ is defined as

$$E_G^{(n)}(F) = E_G^{(1)}\left(E_G^{(n-1)}(F)\right)$$

where $n \geq 1$ is an integer and $E_G^{(0)}(F) = F$.”

“The union performed at each step guarantees that geodesic erosion of an image remains greater than or equal to its mask image.”

Gonzalez and Woods (2018, p. 667)
Geodesic Erosion

Marker, \( F \)

Marker eroded by \( B \)

Mask, \( G \)

Geodesic erosion, \( E_G^{(1)}(F) \)

(This is the eroded marker image masked by \( G \).)
Morphological Reconstruction by Dilation and by Erosion

Segmentation of nontrivial images is one of the most important steps in image processing. Segmentation accuracy determines the eventual success of computerized analysis procedures. For this reason, one must take every practical measure to improve the probability of rugged segmentation. To this end, one might consider the following techniques:...
Grayscale Erosion

- Structuring elements in grayscale morphology are either flat or nonflat.
- The erosion of image $f$ by a flat structuring element $b$ at location $(x,y)$ is defined as the minimum value of the image in the region coincident with $b(x,y)$ when the origin of $b$ is at $(x,y)$:

$$[f \ominus b](x, y) = \min_{(s, t) \in b} \{f(x + s, y + t)\}$$

Gonzalez and Woods (2018, p. 674)
The grayscale dilation of image $f$ by a flat structuring element $b$ at location $(x,y)$ is defined as the maximum value of the image in the region coincident with $b(x,y)$ when the origin of $b$ is at $(x,y)$:

$$[f \oplus b](x,y) = \max_{(s,t) \in \hat{b}} \{ f(x-s, y-t) \}$$

where

$$\hat{b}(c, d) = b(-c, -d)$$

Gonzalez and Woods (2018, p. 674)
Grayscale Dilation and Erosion

Original image  Erosion using flat disk SE of radius 2  Dilation using the same SE

Gonzalez and Woods (2018, p. 679)
Erosion and Dilation with Nonflat SE

- The erosion of image $f$ by a nonflat structuring element $b_N$ is defined as

$$[f \ominus b_N](x, y) = \min_{(s, t) \in b_N} \{ f(x + s, y + t) - b_N(s, t) \}$$

- The dilation of image $f$ by a nonflat structuring element $b_N$ is defined as

$$[f \oplus b_N](x, y) = \max_{(s, t) \in \hat{b}_N} \{ f(x - s, y - t) + \hat{b}_N(s, t) \}$$

Gonzalez and Woods (2018, p. 679)
Grayscale Opening and Closing

- The grayscale opening of image $f$ by structuring element $b$ is defined as

$$f \circ b = (f \ominus b) \oplus b$$

- The grayscale closing of image $f$ by structuring element $b$ is defined as

$$f \bullet b = (f \oplus b) \ominus b$$

Gonzalez and Woods (2018, p. 680)
Opening of $f$ by $b$ can be interpreted as pushing SE up from below the curve and the top of the peaks are clipped by opening.

Closing of $f$ by $b$ can be interpreted as pushing SE down on top of the curve and the troughs are clipped by closing.

Gonzalez and Woods (2018, p. 680)
“Opening suppresses bright details smaller than SE while leaving dark details unaffected. Closing has the opposite effect.”

Gonzalez and Woods (2018, p. 682)
(A) Cygnus loop supernova taken in X-ray band by NASA’s Hubble.
(B)-(D) Results after opening and then closing using SE of radii 1, 3, 5, respectively.

Gonzalez and Woods (2018, p. 683)
Morphological Gradient

Gradient of image $f$ can be obtained by dilation and erosion used in combination with image subtraction.

$$g = (f \oplus b) - (f \ominus b)$$

Gonzalez and Woods (2018, p. 683)
Top-Hat and Bottom-Hat Transformations

- Top-hat transformation is defined as

\[ T_{\text{hat}}(f) = f - (f \cdot b) \]

- Bottom-hat transformation is defined as

\[ B_{\text{hat}}(f) = (f \cdot b) - f \]

- “Top-hat transformation is used for light objects on a dark background, and the bottom-hat transformation is used in the opposite situation.”

- “Top-hat transformation can be used to correct the effect of non-uniform illumination.”

Gonzalez and Woods (2018, p. 684)
Top-Hat Removes Non-uniform Illumination

Original image

Thresholded image

Opening using SE of radius 40

Top-hat transformation

Thresholded top-hat image

Gonzalez and Woods (2018, p. 685)
The image intensity can be interpreted as topography.

In this 3D view, we consider 3 types of points

1. Points belonging to a regional minimum
2. Points at which a drop of water would fall with certainty to a single minimum
3. Points at which water would be equally likely to fall to more than one minimum

For a regional minimum, the set of points satisfying (2) is called the catchment basin or watershed of that minimum.

Points satisfying (3) form watershed lines.

Gonzalez and Woods (2018, p. 787)
Watershed Segmentation

(A) Original image. (B) Topographic view. (C) First stage of flooding. Water covers only the background. (D) Second stage of flooding. Water fills the first basin.

Gonzalez and Woods (2018, p. 788)
(E) Third stage of flooding. Water fills the second basin.
(F) Water begins to merge from 2 basins. A short dam was built.
(G) Longer dams were built.
(H) Final watershed lines superimposed on the original image

Gonzalez and Woods (2018, p. 789)
After the second dilation, water from the two basins merges. So, a 1-pixel-thick dam must be constructed to prevent water to cross over.

Gonzalez and Woods (2018, p. 789)
Watershed Segmentation

(A) Original image.  
(B) Gradient image.  
(C)-(D) Watershed lines superimposed on gradient and original images.

Gonzalez and Woods (2018, p. 789)
Use of Markers

Watershed segmentation typically leads to over-segmentation. Using markers can prevent over-segmentation.

Gonzalez and Woods (2018, p. 789)
Four principal Boolean operations for binary images are AND, OR, XOR, NOT.
(A, B) Binary images obtained by thresholding polarized light images of a thin section of a sandstone.
(C) The result of OR operations of six such images.
(D) Binary image produced by thresholding the grayscale image obtained by combining the same six color images to keep the brightest pixel at each location.

Russ and Neal (2016, p. 441)
Multichannel thresholding and Boolean operations of satellite images can be used to perform ground-use classification.

(A) Visible light image. (B) Ground-use classification. (C) Thresholded mask indicating built-up areas. (D) Change in ground-use over time.

Russ and Neal (2016, p. 441)
A binary image can be used as a mask to erase the background from a continuous-tone image.

Russ and Neal (2016, p. 447)
The Boolean AND operation is useful for applying measurement templates to images.

Consider an example of measuring a layer thickness.

(a) Paint layer viewed in cross-section. (b) Thresholded layer with superimposed grid of vertical lines. (c) AND of lines with layer producing line segments for measurements.

Russ and Neal (2016, p. 452)
Measurement Grids

(a) A cross-section of vein in tissue. (b) Thresholded wall with superimposed grid of radial lines. (c) AND of lines with wall producing line segments for measurement.

Russ and Neal (2016, p. 452)
Example of measuring the speed of a vehicle:
(a) One frame from a traffic camera.
(b) Difference of two sequential images.
(c) AND of (b) with a line

Russ and Neal (2016, p. 453)
“After the pixel groupings are identified as labels, it is possible to carry out Boolean logic at the feature level, rather than at the pixel level.”

Below is a schematic illustration of feature-based AND:
(a) Marker image. (b) Test image. (c) Pixel-based AND selects just the red pixels. (d) Feature-based logic in which markers in (a) select entire features in (b). (e) Feature-based logic in which markers in (b) select entire features in (a)
(a) Original image of cells with stained nuclei.
(b) Nuclei thresholded based on green intensity.
(c) Feature-based AND of (a) and (b) only selects cells containing green-stained nuclei.
(d) Outlines of features from (d) superimposed on (a)

Russ and Neal (2016, p. 455)
References