Filter Design by Pole-zero Placement

- A design of resonator, notch filter, and comb filter can be accomplished by gain matching and pole-zero placement.

Resonator

- A bandpass filter is a filter that passes signals whose frequencies lie within an interval \([F_0,F_1]\).
- When the width of the passband is small in comparison with \(f_s\), the filter is called a narrowband filter.
- A limiting case of a narrowband filter is a filter designed to pass a single frequency \(0 < F_0 < f_s / 2\).
- Such a filter is called a resonator with a resonant frequency \(F_0\).

Schilling and Harris (2012, p. 504)
The frequency response of an ideal resonator is

\[ H_{\text{res}}(f) = \delta_1(f - F_0), \quad 0 \leq f \leq f_s / 2 \]

where \( \delta_1(f) = \begin{cases} 1, & f = 0 \\ 0, & f \neq 0 \end{cases} \)

A simple way to design a resonator is to place a pole near the point on the unit circle that corresponds to the resonant frequency \( F_0 \).

Angle corresponding to frequency \( F_0 \) is \( \theta_0 = 2\pi F_0 / f_s \).

For the filter to be stable, the pole must be inside the unit circle.
Resonator

- For the coefficients of the denominator of $H_{\text{res}}(z)$ to be real, complex poles must occur in conjugate pairs.
- By placing zeros at $z = 1$ and $z = -1$, the resonator will completely attenuates the two end frequencies, $f = 0$ and $f = f_s/2$.
- These constraints yields a resonator transfer function as

$$H_{\text{res}}(z) = \frac{b_0(z - 1)(z + 1)}{\left[ z - r \exp(j\theta_0) \right]\left[ z - r \exp(-j\theta_0) \right]} = \frac{b_0(z^2 - 1)}{z^2 - 2r \cos(\theta_0) + r^2}$$

Schilling and Harris (2012, p. 505)
Let $\Delta F$ denote the radius of the 3 dB passband of filter. That is $|H_{\text{res}}(f)| \geq 1/\sqrt{2}$ for frequency $f$ in the range $[F_0 - \Delta F, F_0 + \Delta F]$.

The pole radius $r$ can be estimated as $r \approx 1 - \pi \Delta F / f_s$.

The gain factor $b_0$ ensures that the passband gain is one.

At the center of the passband $z = z_0 = \exp\left(j2\pi F_0 T\right)$.

Setting $|H(z_0)| = 1$ and solving for $b_0$ yields

$$b_0 = \frac{\left|\exp\left(j2\theta_0\right) - 2r \cos(\theta_0) \exp\left(j\theta_0\right) + r^2\right|}{\left|\exp\left(j2\theta_0\right) - 1\right|}$$

The transfer function is $H_{\text{res}}(z) = \frac{b_0(1 - z^{-2})}{1 - 2r \cos(\theta_0) z^{-1} + r^2 z^{-2}}$.

Schilling and Harris (2012, p. 505-506)
Let’s design a resonator with $F_0 = 200$ Hz, $\Delta F = 6$ Hz, and $f_s = 1200$ Hz.

- The pole angle is $\theta_0 = \frac{2\pi(200)}{1200} = \frac{\pi}{3}$
- The pole radius is $r = 1 - \frac{6\pi}{1200} = 0.9843$
- The gain factor is $b_0 = 0.0156$
- The resonator transfer function becomes

$$H_{res}(z) = \frac{0.0156(1 - z^{-2})}{1 - 0.9843z^{-1} + 0.9688z^{-2}}$$

Schilling and Harris (2012, p. 506)
Resonator: Example

Pole-zero Plot

Magnitude Response

Schilling and Harris (2012, p. 507)
A notch filter is designed to remove a single frequency $F_0$ called the notch frequency.

The frequency response of an ideal notch filter is

$$H_{\text{notch}}(f) = 1 - \delta_1(f - F_0), \quad 0 \leq f \leq f_s / 2$$

“To design a notch filter, we place a zero at the point on the unit circle corresponding to the notch frequency $F_0$.”

Since $z = \exp(j \theta)$ and the angle corresponding to $F_0$ is $\theta_0 = 2\pi F_0 / f_s$, placing a zero at $z_0 = \exp(j 2\pi F_0 T)$ ensures that $H_{\text{notch}}(F_0) = 0$.

To control the 3 dB bandwidth of the stopband, we place a pole at the same angle with a radius a bit smaller than 1, i.e., $0 \ll r < 1$.

Schilling and Harris (2012, p. 508)
Notch Filter

- To obtain real filter coefficients, both poles and zeros must occur in complex conjugate pairs.
- So, the transfer function of notch filter is

\[
H_{\text{notch}}(z) = \frac{b_0 \left[ z - \exp(j\theta_0) \right] \left[ z - \exp(-j\theta_0) \right]}{\left[ z - r \exp(j\theta_0) \right] \left[ z - r \exp(-j\theta_0) \right]}
\]

\[
= \frac{b_0 \left[ 1 - 2\cos(\theta_0)z^{-1} + z^{-2} \right]}{1 - 2r\cos(\theta_0)z^{-1} + r^2z^{-2}}
\]

- The pole radius \( r \) can be estimated as \( r \approx 1 - \pi\Delta F/f_s \).
- Since the passband includes both \( f = 0 \) and \( f = f_s/2 \), the gain factor \( b_0 \) can be obtained by either setting

\[
\left| H_{\text{notch}}(1) \right| = 1 \quad \text{or} \quad \left| H_{\text{notch}}(-1) \right| = 1
\]

Schilling and Harris (2012, p. 508)
Notch Filter

- Setting $|H_{\text{notch}}(1)| = 1$ corresponds to $f = 0$ and leads to
  \[
  b_0 = \frac{|1 - 2r \cos(\theta_0) + r^2|}{2|1 - \cos(\theta_0)|}
  \]

- Setting $|H_{\text{notch}}(-1)| = 1$ corresponds to $f = f_s/2$ and leads to
  \[
  b_0 = \frac{|1 + 2r \cos(\theta_0) + r^2|}{2|1 + \cos(\theta_0)|}
  \]

Schilling and Harris (2012, p. 508)
Let’s design a notch filter with $F_0 = 800$ Hz, $\Delta F = 18$ Hz, and $f_s = 2400$ Hz.

- The angle of zero is $\theta_0 = 2\pi/3$
- The pole radius is $r = 0.9764$
- The gain factor $b_0$ from $|H_{\text{notch}}(1)| = 1$ is $b_0 = 0.9766$
- The transfer function of the notch filter is

$$H_{\text{notch}}(z) = \frac{0.9766 \left( 1 + z^{-1} + z^{-2} \right)}{1 + 0.9764 z^{-1} + 0.9534 z^{-2}}$$

Schilling and Harris (2012, p. 508-509)
Notch Filter: Example

Schilling and Harris (2012, p. 509-510)
A comb filter is a filter that passes DC, a fundamental frequency $F_0$, and its harmonics.

Frequency response of an ideal comb filter of order $n$ is

$$H_{\text{comb}}(f) = \sum_{i=0}^{\text{floor}(n/2)} \delta_1(f - iF_0), \quad 0 \leq f \leq \frac{f_s}{2}, \quad F_0 = \frac{f_s}{n}$$

The transfer function of a comb filter of order $n$ is

$$H_{\text{comb}}(z) = \frac{b_0}{1 - r^n z^{-n}} = \frac{b_0 z^n}{z^n - r^n}$$

The comb filter has $n$ zeros at the origin, and the poles correspond to the $n$ roots of $r^n$ with $r \approx 1$ and $r < 1$ so that it is stable and highly frequency-selective.

Schilling and Harris (2012, p. 510)
The gain factor $b_0$ can be selected such that the passband at $f = 0$ (DC) is one. Setting $|H_{\text{comb}}(1)| = 1$ and solving for $b_0$ yields $b_0 = 1 - r^n$. 

$n = 10$, $r = 0.9843$, $f_s = 200 \text{ Hz}$, $\Delta F = 1 \text{ Hz}$

Schilling and Harris (2012, p. 511-512)
Inverse Comb Filter

- An inverse comb filter removes DC, a fundamental notch frequency $F_0$, and its harmonics.
- Frequency response of an ideal inverse comb filter of order $n$ is

$$H_{\text{inv}}(f) = 1 - \sum_{i=0}^{\text{floor}(n/2)} \delta_1(f - iF_0), \quad 0 \leq f \leq \frac{f_s}{2}, F_0 = \frac{f_s}{n}$$

- Transfer function of an inverse comb filter of order $n$ is

$$H_{\text{inv}}(z) = \frac{b_0 \left(1 - z^{-n}\right)}{1 - r^n z^{-n}} = \frac{b_0 \left(z^n - 1\right)}{z^n - r^n}$$

- The inverse comb filter has $n$ zeros equally spaced on the unit circle and $n$ equally space poles just inside the unit circle.

Schilling and Harris (2012, p. 511)
Inverse Comb Filter

- Similar to the comb filter, \( r \approx 1 \) and \( r < 1 \).
- The gain factor \( b_0 \) can be selected such that the passband gain at \( f = F_0/2 \) is one where \( F_0 = f_s/n \).
- The point on the unit circle corresponding to \( f = F_0/2 \) is \( z_1 = \exp(j\pi/n) \).
- Setting \( |H_{\text{inv}}(z_1)| = 1 \) yields \( b_0 = \left(1 + r^n\right)/2 \)

\[ n = 11 \]
\[ f_s = 2200 \text{ Hz} \]
\[ \Delta F = 10 \text{ Hz} \]
Applications of Comb Filters

- A comb filter of order $n$ can be used to extract the first $n/2$ harmonics of a noise-corrupted periodic signal with a known fundamental frequency of $F_0$ with $f_s = nF_0$.

- “In astronomy, the astro-comb can increase the precision of existing spectrographs by nearly a hundred fold” (https://en.wikipedia.org/wiki/Astro-comb).

- An inverse comb filter can be used to remove periodic noise corrupting a signal.

Schilling and Harris (2012, p. 513-514)
The tunable plucked-string filter is a simple and effective building block for the synthesis of musical sounds.

“The output from this type of filter can be used to synthesize the sound from stringed instruments such as guitar.”

Schilling and Harris (2012, p. 500)
The design parameters for the plucked-string filter are:

- Sampling frequency $f_s$
- Pitch parameter $0 < c < 1$
- Feedback delay $L$
- Feedback attenuation factor $0 < r < 1$

The block with transfer function $F(z) = \left(1 + z^{-1}\right)/2$ is a first-order lowpass filter.

The block with transfer function $G(z) = \frac{c + z^{-1}}{1 + cz^{-1}}$ is a first-order allpass filter.

The purpose of an allpass filter is to change phase of the input and introduce some delay without changing the magnitude response.

Schilling and Harris (2012, p. 500)
Tunable Plucked-string Filter

- The Z-transform of the summing junction is

\[ E(z) = X(z) + r^L z^{-L} Y(z) = X(z) + r^L z^{-L} G(z) W(z) \]
\[ = X(z) + r^L z^{-L} G(z) F(z) E(z) \]

- Solving for \( E(z) \) yields

\[ E(z) = \frac{X(z)}{1 - r^L z^{-L} G(z) F(z)} \]

Schilling and Harris (2012, p. 500-501)
The Z-transform of the output is

\[ Y(z) = G(z)W(z) = G(z)F(z)E(z) = \frac{G(z)F(z)X(z)}{1 - r^Lz^{-L}G(z)F(z)} \]

The transfer function of the plucked-string filter is

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{G(z)F(z)}{1 - r^Lz^{-L}G(z)F(z)} \]

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Schilling and Harris (2012, p. 500-501)
Tunable Plucked-string Filter

- The Z-transform of the output is

\[ Y(z) = G(z)W(z) = G(z)F(z)E(z) = \frac{G(z)F(z)X(z)}{1 - r^L z^{-L} G(z)F(z)} \]

- The transfer function of the plucked-string filter is

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{G(z)F(z)}{1 - r^L z^{-L} G(z)F(z)} \]

\[ = \frac{0.5 \left[ c + (1 + c)z^{-1} + z^{-2} \right]}{1 + cz^{-1} - 0.5 r^L \left[ cz^{-L} + (1 + c)z^{-(L+1)} + z^{-(L+2)} \right]} \]

- “Plucked-string sound is generated by the filter output when the input is an impulse or a short burst of white noise.”

Schilling and Harris (2012, p. 501)
Tunable Plucked-string Filter

- “The frequency response of the plucked-string filter consists of a series of resonant peaks that gradually decay, depending on the value of $r$.”
- Suppose the desired first resonant frequency is $F_0$.
- Then, $L$ and $c$ can be computed as follows.

$$L = \text{floor} \left( \frac{f_s - 0.5F_0}{F_0} \right)$$

$$\delta = \frac{f_s -(L + 0.5)F_0}{F_0}$$

$$c = \frac{1 - \delta}{1 + \delta}$$

Schilling and Harris (2012, p. 501-502)
Plucked-string Filter: Example

Let $f_s = 44.1$ kHz, $F_0 = 740$ Hz, and $r = 0.999$. Then, we have $L = 59$ and $c = 0.8272$.

Schilling and Harris (2012, p. 502)
**Reference**