FIR Filter Design

Chaiwoot Boonyasiririwat
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Characteristics of FIR Filters

- An \( m \)th-order FIR filter is a discrete-time system with the transfer function:
  \[
  H(z) = b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}
  \]

- The impulse response of an FIR filter is
  \[
  h(k) = b_k, \quad 0 \leq k \leq m
  \]

- FIR filters are always stable since all poles are at \( z = 0 \).

- Linear-phase FIR filters can be obtained using symmetry of the coefficients. A linear-phase filter does not distort a signal in the passband. It only delays the signal by the same amount.

- However, a high-order FIR filter is required to design a frequency-selective filter with a narrow transition band.

Schilling and Harris (2012, p. 406-407)
In some applications, we need to obtain the derivative of an analog signal $x_a(t)$ from its samples $x(k) = x_a(kT)$.

FIR filters can be used as a differentiator.

In real-time applications only the current and past samples can be used so we can one-sided finite difference (FD) to approximate, e.g., the first derivative:

- **First-order FD**:
  \[ y_1(k) = \frac{x(k) - x(k-1)}{T} \]

- **Second-order**
  \[ y_2(k) = \frac{3x(k) - 4x(k-1) + x(k-2)}{2T} \]

Schilling and Harris (2012, p. 407-408)
Signal-to-noise Ratio

- Numerical differentiation is sensitive to noise.
- Consider the noise-corrupted signal \( y(k) = x(k) + v(k) \) where \( v(k) \) is white noise uniformly distributed over the interval \([-c, c]\).
- To specify the size of noise relative to the size of signal, the **signal-to-noise ratio** is defined as

\[
\text{SNR}(y) \overset{\text{def}}{=} 10 \log_{10} \left( \frac{P_x}{P_v} \right) \text{ dB}
\]

where \( P_x \) and \( P_v \) are the average power of \( x \) and \( v \).

- **Example:** Average power of sinusoid of amplitude \( A \) is \( A^2/2 \) while average power of white noise is \( P_v = c^2/3 \). If \( A = 1 \) and \( c = 0.01 \), SNR of noisy sinusoid is 32.22 dB.

Schilling and Harris (2012, p. 409)
Numerical Differentiator: Example

Differentiation of noise-free signal

Differentiation of noise-corrupted signal

Schilling and Harris (2012, p. 409-410)
Signal-to-noise Ratio

“Typically, additive white noise $v(k)$ has zero mean and is statistically independent of the signal $x(k)$, which means that $E[x(k)v(k)] = E(x(k)) E[v(k)]$.”

The average power of the signal $y(k)$ corrupted with zero-mean white noise $v(k)$ and the average power of the noise-free signal $x(k)$ are related by

$$P_y = E\left[y^2(k)\right] = E\left[\left\{x(k) + v(k)\right\}^2\right]$$

$$= E\left[x^2(k) + 2x(k)v(k) + v^2(k)\right]$$

$$= E\left[x^2(k)\right] + 2E\left[x(k)\right]E\left[v(k)\right] + E\left[v^2(k)\right]$$

$$= P_x + P_v, \quad E\left[v(k)\right] = 0$$

Schilling and Harris (2012, p. 411)
Recall that the frequency response can be written as

$$H(f) = A_r(f) \exp\left[j\left(\alpha - \pi mfT\right)\right]$$

where $A_r(f)$ is called the **amplitude response** which is real but can be positive or negative.

“For $m$th-order linear-phase FIR filters, the design specifications are formulated in terms of the desired amplitude response $A_r(f)$.”

Schilling and Harris (2012, p. 411)
The passband ripple $\delta_p$ now represents the radius of a region centered about $A_r(f) = 1$.

The stopband attenuation $\delta_s$ now is the radius of a region centered about $A_r(f) = 0$. 

Schilling and Harris (2012, p. 411)
“The basic idea of windowing method is to truncate the desired impulse response to a finite number of samples.”

When $h(k)$ is noncausal, the DTFT of $h(k)$ gives the frequency response

$$H(f) = \sum_{k=-\infty}^{\infty} h(k) \exp(-j2\pi kfT)$$

which is periodic with period $f_s$.

The $k$th Fourier coefficient is

$$h(k) = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} H(f) \exp(j2\pi kfT) \, df, \quad -\infty < k < \infty$$

Schilling and Harris (2012, p. 412)
Truncated Impulse Response

Type-1 and type-2 filters:

- Let \( m \) be the filter order. \( h(k) \) exhibits even symmetry about \( k = m/2 \). The desired frequency response for a type-1 or type-2 linear-phase FIR filter with a group delay of \( \tau = mT/2 \) is

\[
H(f) = A_r(f) \exp(-j\pi fmT)
\]

where the amplitude response \( A_r(f) \) is a real even function specified by the filter designer.

- The impulse response then becomes

\[
h(k) = 2T \int_0^{f_s/2} A_r(f) \cos \left[ 2\pi (k - 0.5m)fT \right] df, \quad 0 \leq k \leq m
\]

Schilling and Harris (2012, p. 412-413)
Truncated Impulse Response

Type-3 and type-4 filters:

- The desired frequency response for a type-3 or type-4 linear-phase FIR filter with a group delay of $\tau = mT/2$ is
  \[
  H(f) = jA_r(f) \exp(-j\pi mfT)
  \]
  where $A_r(f)$ is a real odd function specified by the filter designer.

- The impulse response then becomes
  \[
  h(k) = -2T \int_0^{f_s/2} A_r(f) \sin \left[ 2\pi (k - 0.5m) fT \right] df, \quad 0 \leq k \leq m
  \]
Truncated Impulse Response

Impulse response of ideal frequency-selective linear-phase type-1 filters of order \( m = 2p \).

<table>
<thead>
<tr>
<th>Filter</th>
<th>( h(p) )</th>
<th>( h(k), 0 \leq k \leq m, k \neq p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lowpass</td>
<td>( 2F_0 T )</td>
<td>( \frac{\sin[2\pi(k - p)F_0 T]}{\pi(k - p)} ) – ( \frac{\sin[2\pi(k - p)F_0 T]}{\pi(k - p)} )</td>
</tr>
<tr>
<td>Highpass</td>
<td>( 1 - 2F_0 T )</td>
<td>( \frac{\sin[2\pi(k - p)F_1 T]}{\pi(k - p)} ) – ( \frac{\sin[2\pi(k - p)F_1 T]}{\pi(k - p)} )</td>
</tr>
<tr>
<td>Bandpass</td>
<td>( 2(F_1 - F_0) T )</td>
<td>( \frac{\sin[2\pi(k - p)F_0 T]}{\pi(k - p)} ) – ( \frac{\sin[2\pi(k - p)F_1 T]}{\pi(k - p)} )</td>
</tr>
<tr>
<td>Bandstop</td>
<td>( 1 - 2(F_1 - F_0) T )</td>
<td>( \frac{\sin[2\pi(k - p)F_0 T]}{\pi(k - p)} ) – ( \frac{\sin[2\pi(k - p)F_1 T]}{\pi(k - p)} )</td>
</tr>
</tbody>
</table>

Schilling and Harris (2012, p. 414)
Consider a problem of designing a lowpass filter with cutoff frequency $F_0 = f_s/4$ where $f_s = 100$ Hz.

Suppose $m = 2p = 40$ is used to approximate $H(f)$.

Then, $p = 20$ and the filter coefficients are

$$h(p) = 2F_0T = 2 \frac{f_s}{4} \frac{1}{f_s} = 0.5$$

and

$$h(k) = \frac{\sin \left[ 2\pi(k - p)F_0T \right]}{\pi(k - p)} = 0.5 \sin c \left[ 0.5(k - p) \right], \quad 0 \leq k \leq m$$

The delay in this case is $\tau = pT = 0.2$ s.
The impulse response of this filter has even symmetry and is centered at $k = 20$. 

Schilling and Harris (2012, p. 415)
The Gibb’s phenomenon occurs at the cutoff frequency.

Discontinuities due to sign change in amplitude response where \( A(f) = 0 \).

Phase wrap around

Discontinuities due to sign change in amplitude response where \( A(f) = 0 \).

Schilling and Harris (2012, p. 415)
When the desired magnitude response contains discontinuities, there will be oscillations in the magnitude response due to the truncation of the impulse response – the Gibb’s phenomenon.

These oscillations cause the passband ripple and stopband attenuation to be large.

The oscillations can be reduced at the expense of increasing the width of the transition band.

“To see how this can be done, first notice that the transfer function of a filter can be written as”

\[
H(z) = \sum_{i=-\infty}^{\infty} w_R(i)h(i)z^{-i}
\]

Schilling and Harris (2012, p. 416)
Windowing

“This corresponds to a causal filter of order \( m \) when \( w_R(i) \) is the rectangular window of order \( m \):”

\[
    w_{R}(i) \overset{\text{def}}{=} \begin{cases} 
        1, & 0 \leq i \leq m \\
        0, & \text{otherwise}
    \end{cases}
\]

“Thus, truncation of the impulse response to \( 0 \leq i \leq m \) is equivalent to multiplication of the impulse response by a rectangular window.”

“The abrupt truncation of the impulse response causes the oscillations associated with the Gibb’s phenomenon. The amplitude of the oscillations can be decreased by \textit{tapering} the impulse response to zero gradually.”

Schilling and Harris (2012, p. 416)
“Recall that for an FIR filter the numerator coefficients are $b_i = h(i)$. If $w(i)$ is a window of order $m$, then the tapered filter coefficients are $b_i = w(i)h(i)$, $0 \leq i \leq m$.”

<table>
<thead>
<tr>
<th>Type</th>
<th>Name</th>
<th>$w(i)$, $0 \leq i \leq m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Rectangular</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>Hanning</td>
<td>.5 $- .5 \cos \left( \frac{\pi i}{5m} \right)$</td>
</tr>
<tr>
<td>2</td>
<td>Hamming</td>
<td>.54 $- .46 \cos \left( \frac{\pi i}{5m} \right)$</td>
</tr>
<tr>
<td>3</td>
<td>Blackman</td>
<td>.42 $- .5 \cos \left( \frac{\pi i}{5m} \right) + .08 \cos \left( \frac{2\pi i}{5m} \right)$</td>
</tr>
</tbody>
</table>

Schilling and Harris (2012, p. 416-417)
Windowed Lowpass Filter: Example

- Consider the lowpass filter design again.
- Using the rectangular windows, the filter coefficients are
  \[ b_i = 0.5 \text{sinc}[0.5(i - p)], \quad 0 \leq i \leq m \]
- Using the Hann windows, the filter coefficients are
  \[ b_i = 0.25\left[1 + \cos(\pi i/p)\right] \text{sinc}[0.5(i - p)], \quad 0 \leq i \leq m \]
- Using the Hamming windows, the filter coefficients are
  \[ b_i = 0.5\left[0.54 + 0.46\cos(\pi i/p)\right] \text{sinc}[0.5(i - p)], \quad 0 \leq i \leq m \]
- Using the Blackman windows, the filter coefficients are
  \[ b_i = 0.5\left[0.42 + 0.5\cos(\pi i/p) + 0.08\cos(2\pi i/p)\right] \text{sinc}[0.5(i - p)] \]
Windowed Lowpass Filter: Example

Lowpass magnitude response using **rectangular window** with $m = 40$

![Lowpass Filter Using Rectangular Window](image)

Schilling and Harris (2012, p. 418)
Windowed Lowpass Filter: Example

Lowpass magnitude response using **Hann window** with $m = 40$

Schilling and Harris (2012, p. 418)
Windowed Lowpass Filter: Example

Lowpass magnitude response using Hamming window with $m = 40$

Schilling and Harris (2012, p. 419)
Windowed Lowpass Filter: Example

Lowpass magnitude response using **Blackman window** with $m = 40$

Schilling and Harris (2012, p. 419)
Windowing

“The improvements in passband ripple and stopband attenuation achieved by using windowing come at the cost of a wider transition band. However, the width of the transition band for each window can be controlled by the filter order $m$."

| Window Type   | $\hat{B} = |F_s - F_p|/f_s$ | $\delta_p$ | $\delta_s$ | $A_p$ (dB) | $A_s$ (dB) |
|---------------|-----------------------------|------------|------------|------------|------------|
| Rectangular   | $\frac{.9}{m}$              | .0819      | .0819      | .742       | 21         |
| Hanning       | $\frac{3.1}{m}$              | .0063      | .0063      | .055       | 44         |
| Hamming       | $\frac{3.3}{m}$              | .0022      | .0022      | .019       | 53         |
| Blackman      | $\frac{5.5}{m}$              | .00017     | .00017     | .0015      | 75.4       |

Schilling and Harris (2012, p. 420)
“The Kaiser window $w_K(i)$ is a near-optimal window based on the zeroth-order modified Bessel functions of the first kind.”

$$w_K(i) = \frac{I_0\left(\beta \sqrt{1 - \left(\frac{i-p}{p}\right)^2}\right)}{I_0(\beta)}, \quad 0 \leq i \leq m$$

Here, $I_0(x)$ is a zeroth-order modified Bessel function of the first kind which can be evaluated in MATLAB using the function `besseli(0,x)`.

“`The Kaiser windows has a shape parameter $\beta \geq 0$, that allows the user to control the trade-off between the main lobe width and the side lobe amplitudes.”

Schilling and Harris (2012, p. 420)
Kaiser Window

- When $\beta = 0$, Kaiser window = rectangular window.
- “For a given window size $m$, increasing leads to an increase in the stopband attenuation $A_s$ but the main lobe also get wider, thereby increasing the width of the transition band.”
- “The width of the transition band can also be reduced by increasing $m$.”

Schilling and Harris (2012, p. 420)
Kaiser Window

“Kaiser (1974) developed the following approximations for determining suitable values for $\beta$ and $m$, given a desired stopband attenuation $A_s$ and a desired normalized transition band $\hat{B} = \left| F_s - F_p \right|/f_s$.”

\[
\beta \approx \begin{cases} 
0.1102(A_s - 8.7), & A_s > 50 \\
0.5842(A_s - 21)^4 + 0.07886(A - 21), & 21 \leq A_s \leq 50 \\
0, & A_s < 21 
\end{cases}
\]

\[
m \approx \frac{A_s - 8}{4.568\pi \hat{B}}
\]
Windowed FIR Filter Design Algorithm

1. Pick $m > 0$ and a window function $w(i)$.
2. For $i = 0$ to $m$ compute
   \[
   b_i = \begin{cases}
   w(i)2T \int_0^{f_s/2} A_r(f) \cos \left[ 2\pi (i - 0.5m) fT \right] df, & \text{type 1/2} \\
   -w(i)2T \int_0^{f_s/2} A_r(f) \sin \left[ 2\pi (i - 0.5m) fT \right] df, & \text{type 3/4}
   \end{cases}
   \]
3. Set
   \[
   H(z) = \sum_{i=0}^{m} b_i z^{-i}
   \]

Schilling and Harris (2012, p. 421)
Let’s design a bandpass filter with cutoff frequencies $F_0 = f_s/8$ and $F_1 = 3f_s/8$ using the Blackman window:

$$w(i) = 0.42 - 0.5 \cos\left(\frac{\pi i}{0.5m}\right) + 0.08 \cos\left(\frac{2\pi i}{0.5m}\right)$$

| Window Type  | $\hat{B} = |F_s - F_p|/f_s$ | $\delta_p$ | $\delta_s$ | $A_p$ (dB) | $A_s$ (dB) |
|--------------|-----------------------------|------------|------------|------------|------------|
| Blackman     | $\frac{5.5}{m}$             | .00017     | .00017     | .0015      | 75.4       |

Using the previous algorithm, the filter coefficient are $b_p = 0.5$ and for $0 \leq i \leq m$, $i \neq p$,

$$b_i = \frac{w(i)\left\{\sin[0.75\pi(i - p)] - \sin[0.25\pi(i - p)]\right\}}{\pi(i - p)}$$

Schilling and Harris (2012, p. 421)
Windowed Bandpass Filter: Example

\[ m = 80 \]

Schilling and Harris (2012, p. 421)
Frequency Sampling Method

- Frequency-sampling method is another technique for designing a linear-phase FIR filter using samples of the desired frequency response.
- “Suppose there are \( N \) frequency samples uniformly distributed over the range \( 0 \leq f \leq f_s \) with the \( i \)th discrete frequency being”
  \[
f_i = if_s / N, \quad 0 \leq i < N
\]
- “Recall that the samples of the frequency response of an FIR filter can be obtained directly from the DFT of the impulse response.”

Schilling and Harris (2012, p. 423)
“For an FIR filter of order $m = N - 1$, we have $H(f_i) = H(i)$ for $0 \leq i \leq N$ where $H(i) = \text{DFT}\{h(k)\}$.”

Therefore,

$$h(k) = \text{IDFT}\left\{H(f_i)\right\}, \quad 0 \leq k < N$$

which is a filter whose frequency response $H(f)$ interpolates the $N$ samples.

Suppose $h(k)$ is the impulse response of a type-1 or type-2 linear-phase FIR filter. Then its frequency response is

$$H(f) = A_r(f) \exp(-j\pi mfT)$$

Schilling and Harris (2012, p. 424)
Then the corresponding impulse response is

\[ h(k) = \frac{1}{N} \sum_{i=0}^{N-1} H(f_i) \exp\left(j2\pi ik/N\right) \]

\[ = \frac{1}{N} \sum_{i=0}^{N-1} A_r(f_i) \exp\left[j2\pi i(k - 0.5m)/N\right] \]

For real \( h(k) \), the sine terms cancel one another and we obtain

\[ h(k) = \frac{A_r(f_0)}{N} + \frac{1}{N} \sum_{i=1}^{N-1} A_r(f_i) \cos\left[2\pi i(k - 0.5m)/N\right] \]

Schilling and Harris (2012, p. 424)
“Using the symmetry properties of DFT, one can show that the contributions of the \( i \) term and the \( N - i \) term are identical which means they can be combined.”

“Recall that \( b_k = h(k) \) for an FIR filter.”

We then obtain the coefficient of a linear-phase frequency-sampled filter of order \( m \) where \( m = N - 1 \):

\[
b_k = \frac{A_r(0)}{m + 1} + \frac{2}{m + 1} \sum_{i=1}^{m/2} A_r(f_i) \cos \left( \frac{2\pi i (k - 0.5m)}{m + 1} \right), \quad 0 \leq k \leq m
\]

When \( m \) is odd, use the floor function to compute the upper limit of the sum \( m/2 \) to get an integer.
Let’s design a lowpass filter of order \( m = 20 \) with cutoff frequency \( F_0 = f_s/4 \). In this case, the samples of the desired amplitude response are

\[
A_r(f_i) = \begin{cases} 
1, & 0 \leq i \leq 5 \\
0, & 6 \leq i \leq 10
\end{cases}
\]

Using the formula in the previous slide, we obtain

\[
b_k = \frac{1}{21} + \frac{2}{21} \sum_{i=1}^{5} \cos \left( \frac{2\pi i(k-10)}{21} \right), \quad 0 \leq k \leq 20
\]

Schilling and Harris (2012, p. 425)
Frequency-sampled Lowpass Filter

Schilling and Harris (2012, p. 426)
The ringing in the magnitude response in the previous example is caused by the abrupt transition from passband to stopband in the desired magnitude response.

These oscillations can be reduced by tapering the filter coefficients using a windows function but the transition band will be wider.

With the frequency sampling method, explicitly specifying $A(f)$ in the transition band by including transition-band samples will widen the transition band but has the effect of improving the passband ripple and stopband attenuation.

Schilling and Harris (2012, p. 425)
Consider the same lowpass filter design problem but in this case we insert a single transition band sample

\[
A_r (f_i) = \begin{cases} 
1, & 0 \leq i \leq 5 \\
0.5, & i = 6 \\
0, & 7 \leq i \leq 10 
\end{cases}
\]

In this case, the filter coefficients are

\[
b_k = \frac{1}{21} + \frac{2}{21} \left\{ \sum_{i=1}^{5} \cos \left[ \frac{2\pi i (k - 10)}{21} \right] + 0.5 \cos \left[ \frac{2\pi 6 (k - 10)}{21} \right] \right\}
\]

Schilling and Harris (2012, p. 425)
Filter with Transition-band Sample

Schilling and Harris (2012, p. 426)
Filter with Optimal Transition-band Sample

- Consider the same lowpass filter design problem but in this case we insert a general transition band sample

$$A_r (f_i) = \begin{cases} 
1, & 0 \leq i \leq 5 \\
x, & i = 6 \\
0, & 7 \leq i \leq 10 
\end{cases}$$

- In this case, the filter coefficients are

$$b_k(x) = \frac{1}{21} + \frac{2}{21} \left\{ \sum_{i=1}^{5} \cos \left[ \frac{2\pi i(k-10)}{21} \right] + x \cos \left[ \frac{2\pi 6(k-10)}{21} \right] \right\}$$

- We want to find the value of $x$ that maximizes the stopband attenuation $A_s$.

Schilling and Harris (2012, p. 425)
We can achieve this by computing the stopband attenuation \( A_s \) for 3 distinct values of \( x \), e.g., \([0.25, 0.5, 0.75]\) which are in the range \( 0 < x < 1 \).

The quadratic polynomial passes through these 3 data points is \( A_s(x) = c_1 + c_2x + c_3x^2 \).

The coefficients \( c_i \) must satisfy the linear system

\[
\begin{bmatrix}
1 & x_1 & x_1^2 \\
1 & x_2 & x_2^2 \\
1 & x_3 & x_3^2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} =
\begin{bmatrix}
A_s(x_1) \\
A_s(x_2) \\
A_s(x_3)
\end{bmatrix}
\]

After solving this system to obtain the value of \( c_i \), we can differentiate the quadratic model to find \( x \) that maximizes \( A_s(x) \).

Schilling and Harris (2012, p. 428)
Filter with Optimal Transition-band Sample

- We then obtain the optimal transition sample value as
  \[ x_{\text{max}} = -\frac{c_2}{2c_3} \]

- In this case, we have \( c_1 = 18.35, c_2 = 62.74, c_3 = -80.83 \).

- Thus, the optimal transition band sample is
  \[ A_r (f_6) = x_{\text{max}} = 0.388 \]

Schilling and Harris (2012, p. 428)
The frequency sampling method can also be used to design linear-phase FIR filters whose impulse response exhibit odd symmetry about $k = m/2$.

This corresponds to a filter with frequency response

$$H(f) = jA_r(f) \exp(-j\pi mfT)$$

In this case, the $k$th sample of the impulse response is

$$h(k) = \frac{1}{N} \sum_{i=0}^{N-1} H(f_i) \exp(j2\pi ik/N)$$

$$= \frac{j}{N} \sum_{i=0}^{N-1} A_r(f_i) \exp(-j\pi mf_i T) \exp[j2\pi ik/N]$$

$$= \frac{j}{N} \sum_{i=0}^{N-1} A_r(f_i) \exp\left\{ j\left[2\pi i(k - 0.5m)/N \right] \right\}$$

Schilling and Harris (2012, p. 425)
“For real \( h(k) \) the cosine terms cancel one another and we have”

\[
h(k) = -\frac{1}{N} \sum_{i=0}^{N-1} A_r(f_i) \sin\left[\frac{2\pi i(k - 0.5m)}{N}\right]
\]

“Since \( A_r(f) \) is an odd function, the \( i = 0 \) term drops out because \( A_r(0) = 0 \).”

In this case, the coefficients of a linear-phase frequency-sampled filter of order \( m = N - 1 \) are

\[
b_k = \frac{-2}{m+1} \sum_{i=1}^{\text{floor}(m/2)} A_r(f_i) \sin\left[\frac{2\pi i(k - 0.5m)}{m+1}\right], \quad 0 \leq k \leq m
\]

Schilling and Harris (2012, p. 429)
Least-squares Method

- “The frequency response of a digital filter is periodic with the period $f_s$.”
- “Because the windows method uses a truncated Fourier series expansion of the desired magnitude response $A_d(f)$, this method produces a filter that is optimal in the sense that it minimizes the objective function:”

$$J = \int_0^{f_s/2} \left[ A_d(f) - A_r(f) \right]^2 df$$

- “The actual amplitude response $A_r(f)$ is real but can be positive or negative. For type-1 or type-2 filters, $A_r(f)$ is even, and for type-3 or type-4 filters, $A_r(f)$ is odd.”
- We can design a filter using a discrete version of $J$.

Schilling and Harris (2012, p. 430)
Least-squares Method

- Let \( \{F_0, F_1, \ldots, F_p\} \) be a set of \( p + 1 \) distinct frequencies with \( F_0 = 0, F_p = f_s/2 \), and
  \[
  F_0 < F_1 < \cdots < F_p
  \]
- Let \( w(i) > 0 \) be a weighting function specifying the relative importance of discrete frequency \( F_i \).
- The weighted discrete version of the objective function \( J \) can be formulated as follows.
  \[
  J_p = \sum_{i=0}^{p} w^2(i) \left[ A_r(F_i) - A_d(F_i) \right]^2
  \]
- A filter design technique that minimizes \( J_p \) is called a least-squares method.

Schilling and Harris (2012, p. 430-431)
Consider the type-1 linear-phase filter of order $m \leq 2p$.

\[ H(z) = \sum_{i=0}^{m} b_i z^{-i} \]

Recall that for a type-1 filter, $m$ is even and the impulse response satisfies the even symmetry condition

\[ h(m-k) = h(k) \]

For FIR filters, $b_i = h(i)$ so $b_{m-i} = b_i$ for $0 \leq i \leq m$.

Let $\theta = 2\pi f T$ and $m = 2r$.

The frequency response can be written as

\[ H(f) = \sum_{i=0}^{m} b_i \exp(-ji\theta) = \exp(-jr\theta) \sum_{i=0}^{m} b_i \exp[-j(i-r)\theta] \]

Schilling and Harris (2012, p. 431)
Least-squares Method

- Using the symmetry condition $b_{m-i} = b_i$ and the Euler’s formula, we obtain
  \[ H(f) = A_r(f) \exp(-jr\theta) \]
  where
  \[ A_r(f) = b_r + 2 \sum_{i=0}^{r-1} b_i \cos[(i - r)\theta] \]

- Let’s define $c_i = b_i$ for $i \neq r$ and $c_r = b_r / 2$.

- The amplitude response of a type-1 linear-phase FIR filter of order $2r$ is
  \[ A_r(f) = 2 \sum_{i=0}^{r} c_i \cos[2\pi(i - r)fT] \]

Schilling and Harris (2012, p. 431)
Least-squares Method

- Now we can find an optimal value for $c$ from the objective function

$$J_p(c) = \sum_{i=0}^{p} w^2(i) \left[ 2 \sum_{k=0}^{r} c_k \cos[2\pi(k-r)F_iT] - A_d(F_i) \right]^2$$

- Let the $(p+1)\times(r+1)$ matrix $G$ and $(p+1)$ column vector $d$ be defined as follows.

$$G_{ik} = 2w(i) \cos[2\pi(k-r)F_iT], \quad 0 \leq i \leq p, \quad 0 \leq k \leq r$$

$$d_i = w(i)A_d(F_i), \quad 0 \leq i \leq p$$

- If $c = [c_0, c_1, \ldots, c_r]^T$ is the unknown coefficient vector, the objective function can be written as

$$J_p(c) = (Gc - d)^T (Gc - d)$$

Schilling and Harris (2012, p. 432)
Least-squares Method

- Since \( p \geq r \), this linear system is overdetermined.
- The least-squares solution is the solution of the normal equation

\[
G^T G c = G^T d
\]

- Once the \((r+1)\)-vector \(c\) is determined, the \((m+1)\)-vector \(b\) is obtained as follows.

\[
b = \begin{cases} 
c_i, & 0 \leq i < r \\
2c_r, & i = r \\
c_{2r-i}, & r < i \leq 2r
\end{cases}
\]

Schilling and Harris (2012, p. 432)
Example: Least-squares Bandpass Filter

Let’s design a bandpass filter with a piecewise-linear magnitude response that features a bandpass of

$$3f_s/16 \leq |f| \leq 5f_s/16$$

and a transition band width $f_s/32$.

Suppose $p = 40$ and uniformly spaced discrete frequencies are used:

$$F_i = \frac{if_s}{2p}, \quad 0 \leq i \leq p$$

Two cases are considered:

1. Uniform weighting
2. Weight the passband sample by $w(i) = 10$

In both cases, the FIR filter is of order $m = 40$. 

Schilling and Harris (2012, p. 432)
Example: Least-squares Bandpass Filter

Schilling and Harris (2012, p. 433)
Reference