Introduction to Signal Processing and Sampling and Reconstructing Continuous-time Signals

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What is Signal?

- Signal is a variable or function that contains information of a physical system.

Examples:
- Audio and speech signals
- Image and video signals
- Medical signals: EEG, EKG, ultrasound
- Geophysical signals: earthquakes, tide gauge, LIDAR
- Climate signals
- SONAR, RADAR

[Image: EEG Signal](http://www.intechopen.com/source/html/18890/media/image4.jpeg)
Types of Signals

- **Analog signal**
  - Continuous signal with infinite resolution
  - Typically referred to **electrical signal** converted from a physical variable by a **transducer**, e.g. microphone converting audio signal into electrical (analog) signal.

- **Discrete signal**: signal values are available at some discrete points in space or time

- **Digital signal**: value of discrete signal is stored with a **finite precision**, e.g., in computer as fixed-point or floating-point numbers.
Sampling

A **discrete-time signal** $x(n)$ can be obtained by taking samples of an **analog signal** $x_a(t)$

$$x(n) = x_a(nT), \quad |n| = 0, 1, 2, \ldots$$

where $T$ is the **sampling interval** or time between samples, and sampling frequency or **sampling rate** $f_s = 1/T$ (Hz).

“When finite precision is used to represent the value of $x(n)$, the sequence of quantized values is called a **digital signal**.”

Schilling and Harris (2012; p.3,12)
In exploration seismology, signals are in digital form.

Any system or algorithm which processes input digital signal $x(n)$ and produces an output digital signal $y(n)$ is a **digital signal processor**.
Causal and Acausal Signal

Causal signal: \( x_a(t) = 0 \) for \( t < 0 \)

Acausal signal: \( x_a(t) \neq 0 \) for \( \exists t < 0 \)

Examples of causal signals:

Heaviside step function: \( \mu_a(t) \) defined as:
\[
\mu_a(t) = \begin{cases} 
0, & t < 0 \\
1, & t \geq 0 
\end{cases}
\]

Dirac delta function:
\[
\int_{-\infty}^{t} \delta_a(\tau) d\tau = \mu_a(t), \quad \int_{-\infty}^{\infty} \delta_a(t) dt = 1; \quad \delta_a(t) = 0, t \neq 0
\]

\[
\int_{-\infty}^{\infty} x_a(t) \delta_a(t - t_0) dt = x_a(t_0)
\]

\[\text{Sifting property of delta function}\]

Schilling and Harris (2012, p.15-16)
System Classification

- Continuous system: continuous input and output
- Discrete system: discrete input and output
- Linear system:
  - Time-invariant system: 
  - Bounded signal: 
  - Stable system: every bounded input produces a bounded output (BIBO)

System is considered as a function or operator:

\[ y = S(x) \]

Schilling and Harris (2012, p.16-18)
Magnitude and Phase

- **Forward and inverse Fourier transforms:**

  \[ X_a(f) = F\left\{ x_a(t) \right\} = \int_{-\infty}^{\infty} x_a(t)e^{-i2\pi ft} \, dt \]

  \[ x_a(t) = F^{-1}\left\{ X_a(f) \right\} = \int_{-\infty}^{\infty} X_a(f)e^{i2\pi ft} \, df \]

- **Magnitude spectrum:** \( A_a(f) = |X_a(f)| \)

- **Phase spectrum:** \( \phi_a(f) = \angle X_a(f) \)

- **Polar form:** \( X_a(f) = A_a(f)e^{i\phi_a(f)} \)

- “For real \( x_a(t) \), the magnitude spectrum is even function of \( f \), and the phase spectrum is odd function of \( f \).”

Schilling and Harris (2012, p.18-19)
Filter and Frequency Response

- **Filter** is a system designed to reshape the spectrum of a signal.
- For linear time-invariant (LTI) continuous-time system \( S \) with input \( x_a(t) \) and output \( y_a(t) \), the **frequency response** \( H_a(f) \) is defined as

\[
H_a(f) \overset{\text{def}}{=} \frac{Y_a(f)}{X_a(f)} = A_a(f) \exp[i\phi_a(f)]
\]

where \( A_a(f) \) is magnitude response of \( S \)
\( \phi_a(f) \) is phase response of \( S \)

Schilling and Harris (2012, p.19)
Impulse Response

- **Impulse response** is the system output when the input is the unit impulse (Dirac function)

\[ h_a(t) = S(\delta_a(t)) \]

- It can be shown that \( F\{\delta_a(t)\} = 1 \)
- As a result, when the system input is the unit impulse, the frequency response of the system is

\[ H_a(f) \overset{\text{def}}{=} \frac{Y_a(f)}{X_a(f)} = Y_a(f) = F\{h_a(t)\} \]

Schilling and Harris (2012, p.20)
Ideal low-pass filter with cut-off frequency $B$ has frequency response

$$H_a(f) = \rho_B(f) = \begin{cases} 1, & |f| \leq B \\ 0, & |f| > B \end{cases}$$

Recall that $Y_a(f) = H_a(f)X_a(f)$

So, the frequency component of $x_a(t)$ in the range $[-B, B]$ passes through the filter without distortion.

Schilling and Harris (2012, p.20)
Using the inverse Fourier transform on the frequency response, we obtain the impulse response of this filter as

\[ h_a(t) = 2B \text{sinc}(2\pi B t) \]

where \( \text{sinc}(x) \stackrel{\text{def}}{=} \frac{\sin(x)}{x} \)

Sinc function is an acausal output of the filter when the input is causal. “So, this filter cannot be realized with physical hardware.”

Schilling and Harris (2012, p.20-21)
Periodic impulse train with period $T$ is defined as:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta_a(t - nT)$$

The sampled version of signal $x_a(t)$ denoted by $\hat{x}_a(t)$ is defined as:

$$\hat{x}_a(t) = x_a(t) \delta_T(t)$$

Amplitude modulation

Schilling and Harris (2012, p.21-23)
Sampling as Modulation

We then have

\[ \hat{x}_a(t) = x_a(t)\delta_T(t) \]

\[ = \sum_{n=-\infty}^{\infty} x_a(t)\delta_a(t-nT) \]

\[ = \sum_{n=-\infty}^{\infty} x_a(nT)\delta_a(t-nT); \quad \delta_a(t-t_0) = 0, t \neq t_0 \]

\[ = \sum_{n=-\infty}^{\infty} x(n)\delta_a(t-nT); \quad x(n) = x_a(nT) \]

\[ x(n) \text{ is a discrete-time signal.} \]
Note that \( \hat{x}_a(t) \) is still a continuous-time signal. If \( \hat{x}_a(t) \) is causal, we can apply the Laplace transform to the signal.

\[
X_a(s) = L \{ x_a(t) \} = \int_0^\infty x_a(t) \exp(-st) \, dt
\]

For causal signals, the Fourier transform is the Laplace transform with \( s = i2\pi f \). Therefore, the spectrum of a causal signal can be obtained from its Laplace transform, i.e.,

\[
X_a(f) = X_a(s) \bigg|_{s=i2\pi f}
\]

Schilling and Harris (2012, p.22-23)
Taking the Laplace transform of \( \hat{x}_a(t) \) and then using \( s = i2\pi f \), we then obtain the spectrum of \( \hat{x}_a(t) \), the sampled version of \( x_a(t) \), as

\[
\hat{X}_a(f) = f_s \sum_{n=-\infty}^{\infty} X_a(f - nf_s)
\]

Aliasing formula

where

\[
X_a(f) = F \{ x_a(t) \}
\]

\[
\hat{X}_a(f) = F \{ \hat{x}_a(t) \}
\]
Band-Limited Signal

“A continuous-time signal \( x_a(t) \) is band-limited to bandwidth \( B \) if and only if its magnitude spectrum satisfies

\[
|X_a(f)| = 0 \quad \text{for} \quad |f| > B.
\]

Schilling and Harris (2012, p.23)
The spectrum of the sampled version of a signal is a sum of scaled and shifted spectra of the original signal.

A replicated version of the spectrum of original signal is scaled by \( f_s \), the sampling frequency, and shifted by \( nf_s \) where \( n \) is integer.

Schilling and Harris (2012, p.23)
Aliasing Formula

Spectrum of the original signal

Scaled and shifted replicas of the original spectrum

http://archives.sensorsmag.com/articles/0103/38/fig5_big.gif
The overlap of the replicas of the original spectrum (aliasing) occurs when $f_s \leq 2B$.

The original signal can no longer be exactly reconstructed!

Why?
Aliasing

(a) Magnitude Spectrum of $X_a$

(b) Magnitude Spectrum of Sampled Signal

Schilling and Harris (2012, p.24)
Shannon Sampling Theorem

“Suppose a continuous-time signal $x_a(t)$ is band-limited to $B$ Hz. Let $\hat{x}_a(t)$ denote the sampled version of $x_a(t)$ using impulse sampling with a sampling frequency $f_s$. Then the samples contain all the information necessary to recover the original signal $x_a(t)$ if $f_s > 2B$.”

Schilling and Harris (2012, p.25)
Nyquist frequency is defined as

\[ f_n \overset{\text{def}}{=} \frac{f_s}{2} \]

If \( x_a(t) \) has any frequency component outside of \( f_n \), then in \( \hat{x}_a(t) \) these frequencies get reflected about \( f_n \) and folded back into the range \([-f_n, f_n] \). This is called aliasing.

Schilling and Harris (2012, p.26)
Under- and Over-Sampling

Undersampling: \[ f_s < 2B \]
- Aliasing occurs.
- Original signal **cannot** be reconstructed.

Oversampling: \[ f_s > 2B \]
- No aliasing.
- Original signal can be reconstructed.
Consider the signal $x_a(t) = \sin(2\pi B t), \quad B = 100$

The spectrum of $x_a(t)$ is

$$X_a(t) = \frac{i}{2} \left[ \delta(f + 100) - \delta(f - 100) \right]$$

Adapted from Example 1.5 of Schilling and Harris (2012, p.25)
Example: Aliasing

To avoid aliasing, we need $f_s > 200$ Hz.

Let’s sample $x_a(t)$ at the rate $f_s = 150$ Hz.

In this case, $T = 0.0067$ s and the samples are

$$x(n) = x_a(nT) = \sin(200\pi nT) = \sin(1.3\pi n)$$

$$= \sin(2\pi n - 0.6\pi n)$$

$$= \sin(2\pi n)\cos(0.6\pi n) - \cos(2\pi n)\sin(0.6\pi n)$$

$$= -\sin(0.6\pi n) = -\sin(100\pi nT)$$

Samples of $x_a(t) = \sin(200\pi t)$ are identical to those of $x_b(t) = \sin(100\pi t)$.
Example: Aliasing

\[ x_a(t) \text{ and } x(n) \]

\[ |X_a(f)| \]

\[ \times 10^4 \]

\[ f_s = 150 \]
\[ f_n = 75 \]
Reconstruction of Continuous Signal

- Signal reconstruction is to recover the spectrum $X_a(f)$ from the spectrum $\hat{X}_a(f)$.
- This can be done by passing $\hat{x}_a(t)$ through an ideal lowpass reconstruction filter $H_{\text{ideal}}(f)$ that removes the side bands and rescales the base band.

$$H_{\text{ideal}}(f) = \begin{cases} 
T, & |f| \leq f_n \\
0, & |f| > f_n
\end{cases}$$

$$X_a(f) = H_{\text{ideal}}(f)\hat{X}_a(f)$$

Schilling and Harris (2012, p.27)
Reconstruction of Continuous Signal

\[ H_{\text{ideal}}(f) \equiv \begin{cases} 
T, & |f| \leq f_n \\
0, & |f| > f_n 
\end{cases} \]

- Remove side bands
- Remove side bands
The impulse response of the ideal reconstruction filter is

\[ h_{\text{ideal}}(t) = F^{-1} \left\{ H_{\text{ideal}}(f) \right\} \]

\[ = 2T f_n \text{sinc}(2\pi f_n t) \]

\[ = \text{sinc}(\pi f_s t) \]

The original continuous signal can be perfectly reconstructed by

\[ x_a(t) = F^{-1} \left\{ H_{\text{ideal}}(f) \hat{X}_a(t) \right\} \]

Schilling and Harris (2012, p.28)
Convolution Theorem

Fourier transform of the convolution of \( a(t) \) and \( b(t) \), denoted as \( a(t) * b(t) \), is the pointwise product of their Fourier transforms:

\[
F \{ a(t) * b(t) \} = F \{ a(t) \} F \{ b(t) \} = A(f)B(f)
\]

where

\[
a(t) * b(t) \overset{\text{def}}{=} \int_{-\infty}^{\infty} a(t - \tau) b(\tau) \, d\tau
\]

It can be shown that convolution is commutative, that is

\[
a(t) * b(t) = b(t) * a(t)
\]

https://en.wikipedia.org/wiki/Convolution_theorem
\textbf{Shannon Interpolation Formula}

\[ x_a(t) = F^{-1}\left\{ H_{\text{ideal}}(f)\hat{X}_a(f) \right\} = \int h_{\text{ideal}}(t-\tau)\hat{x}_a(\tau)\,d\tau \]

\[ = \int h_{\text{ideal}}(t-\tau) \left[ \sum_{n=-\infty}^{\infty} x(n)\delta_a(\tau-nT) \right] \,d\tau \]

\[ = \sum_{n=-\infty}^{\infty} x(n) \left[ \int h_{\text{ideal}}(t-\tau)\delta_a(\tau-nT)\,d\tau \right] \]

\[ = \sum_{n=-\infty}^{\infty} x(n) h_{\text{ideal}}(t-nT) \]

\[ = \sum_{n=-\infty}^{\infty} x(n) \text{sinc} \left[ \pi f_s (t-nT) \right] \]

Schilling and Harris (2012, p.28)
Suppose a continuous-time signal $x_a(t)$ is bandlimited to $B$ Hz. Let $x(n) = x_a(nT)$ be the $n$-th sample of $x_a(t)$ using a sampling frequency of $f_s = 1/T$. If $f_s > 2B$, then $x_a(t)$ can be reconstructed from $x(n)$ as follows.

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \text{sinc}\left[\pi f_s (t - nT)\right]$$

The sinc function is used here as a basis function for interpolation.

Schilling and Harris (2012, p.28)
Let $x_a(t)$ be a causal nonzero input to a continuous-time linear system, and let $y_a(t)$ be the corresponding output. The transfer function of the system is defined as

\[
H_a(s) \overset{\text{def}}{=} \frac{Y_a(s)}{X_a(s)}
\]

Since $L\{\delta_a(t)\} = 1$, the transfer function is the Laplace transform of the impulse response

\[
H_a(s) = L\{h_a(t)\}
\]

Schilling and Harris (2012, p.29)
Example: Time Shift

Consider the system \( y_a(t) = x_a(t - \tau) \).

\[
Y_a(s) = \mathcal{L}\{x_a(t - \tau)\} = \int_0^{\infty} x_a(t - \tau) \exp(-st) \, dt
\]

\[
= \int_0^{\infty} x_a(t') \exp[-s(t' + \tau)] \, dt', \quad t' = t - \tau
\]

\[
= \exp(-s\tau) \int_0^{\infty} x_a(t') \exp(-st') \, dt'
\]

\[= \exp(-s\tau) X_a(s)\]

So, \( H_a(s) = \exp(-\tau s) \)

\( H_a(f) = \exp(-i2\pi f \tau) \)

\( H_a(\omega) = \exp(-i\omega \tau) \)

Schilling and Harris (2012, p.29-30)
### Zero-Order Hold

- Exact reconstruction of $x_a(t)$ using the Shannon interpolation formula required an ideal filter which cannot be realized by a physical system.
- The reconstruction can be approximated by a practical filter such as a zero-order hold filter

$$y_a(t) = \int_0^t [x_a(\tau) - x_a(\tau - T)] \, d\tau$$

- The impulse response of a zero-order hold is

$$h_0(t) = \int_0^t [\delta_a(\tau) - \delta_a(\tau - T)] \, d\tau = \mu_a(t) - \mu_a(t-T)$$

Schilling and Harris (2012, p.30)
Zero-Order Hold

- The zero-order hold is linear and time-invariant.
- The response to an impulse of strength \( x(n) \) at time \( t = nT \) will be a pulse of height \( x(n) \) and width \( T \) starting at \( t = nT \).
- When the input is \( \hat{x}_a(t) \), the output will be a piecewise-constant approximation to \( x_a(t) \).
- The transfer function of zero-order hold is:

\[
H_0(s) = \left[ 1 - \exp(-Ts) \right] / s
\]

Schilling and Harris (2012, p.31)
Zero-order hold can be used as a digital-to-analog converter (DAC) while an impulse sampler can be used as an analog-to-digital converter (ADC).

Switch opens and closes every T seconds.

Schilling and Harris (2012, p.31-32)
When a signal that is not bandlimited is sampled, aliasing will occur. To avoid aliasing, a lowpass filter must be applied to the signal. An anti-aliasing filter is a lowpass filter that removes all frequency components outside range $[-f_c, f_c]$, $f_c < f_n = f_s / 2$ where $f_c$ is called the cut-off frequency. The ideal lowpass filter is the optimal choice for an anti-aliasing filter. Butterworth filter is a practical filter that has been widely used as an anti-aliasing filter.

Schilling and Harris (2012, p.33)
“A lowpass Butterworth filter of order $n$ has the magnitude response as follows.”

$$|H_a(f)| = \frac{1}{\sqrt{1 + (f/f_c)^{2n}}}, \quad n \geq 1$$

At the cut-off frequency $f_c$,

$$|H_a(f_c)| = \frac{1}{\sqrt{2}} \quad \text{and} \quad 20 \log_{10} \left\{|H_a(f_c)|\right\} \approx -3 \text{ dB}$$

so $f_c$ is called the 3 dB cutoff frequency of the filter.
Butterworth Filter

The transfer function of a lowpass Butterworth filter of order $n$ is

$$H_n(s) = \frac{\omega_c^n}{s^n + \omega_c a_1 s^{n-1} + \omega_c^2 a_2 s^{n-2} + \cdots + \omega_c^n}, \quad \omega_c = 2\pi f_c$$

“As the order $n$ increases, the magnitude response approaches the ideal lowpass characteristic.”

Schilling and Harris (2012, p.33-34)
First-Order Butterworth Filter

- The transfer function of the first-order Butterworth filter is \( H_1(s) = \frac{\omega_c}{s + \omega_c} \)
- The circuit realization of the first-order Butterworth filter is shown below.
- This circuit requires 3 operational amplifiers, 6 resistors of resistance \( R \), and a capacitor of capacitance \( C \).

\[
\omega_c = \frac{1}{RC}
\]

Schilling and Harris (2012, p.34-35)