Basics of Fluid Mechanics

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Fluids

- A fluid is a substance with no shear strength, i.e., its shear modulus is zero.
- Although a fluid is composed of a large number of molecules, we can approximately treat a fluid as a continuous substance.
- A flow of a fluid is due to an external force such as a pressure difference, gravity, wind, and surface tension.
- External forces can be classified as surface forces and body forces.
- The most important fluid properties are mass density and viscosity.
- Other fluid properties affect fluid flows only under some conditions.

Ferziger et al. (2020)

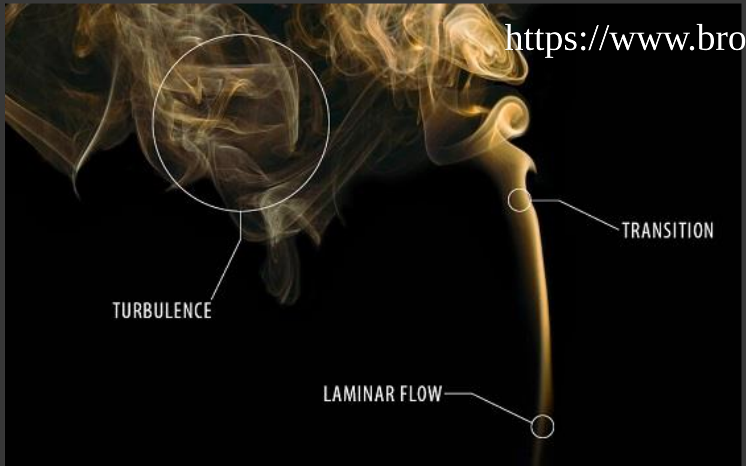
Creeping Flows or Stokes Flows

- The flow speed is low enough that inertia forces are small compared to viscous forces, i.e., the Reynolds number Re << 1.
- The Reynolds number is the ratio of inertial forces to viscous forces.
- The Reynolds number is defined as $\mathrm{Re} = uL/\nu = \rho u L/\mu$ where ρ is the fluid density, u is the flow speed, L is the characteristic length, μ is dynamic viscosity, and $\nu = \mu/\rho$ is kinematic viscosity.
- This flow regime is important in flows with small particles or flows in porous media.
- Creeping flows are governed by the Stokes equations, a linearized, steady-state version of the Navier-Stokes equations.

Ferziger et al. (2020)

Laminar and Turbulent Flows

- Laminar flows: At a larger flow speed such that the inertia is not negligible and fluid particles still follow smooth trajectories.
- In the laminar flow regime the Reynolds number is smaller than a critical value beyond which the flow becomes turbulent: $Re < Re_{critical}$.
- Turbulent flows: When the flow speed is so large that an instability occurs, various random flows could happen.
- Laminar-turbulent transition: A transition from laminar flows to turbulent flows occur when the Reynolds number is in a certain range specific to the situation.

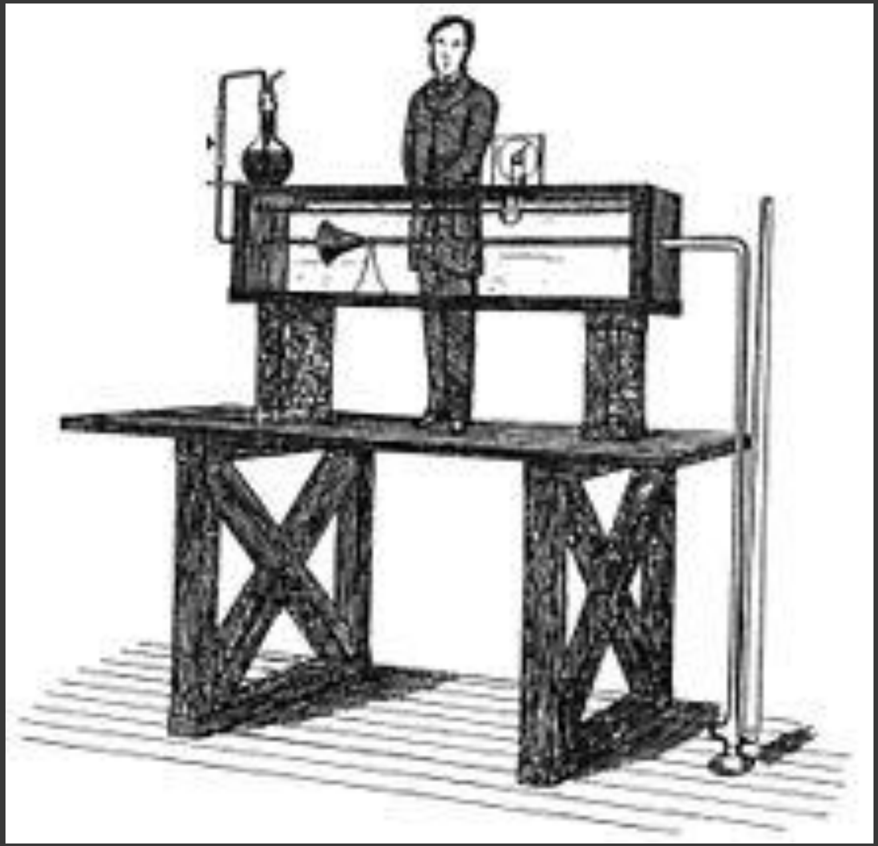


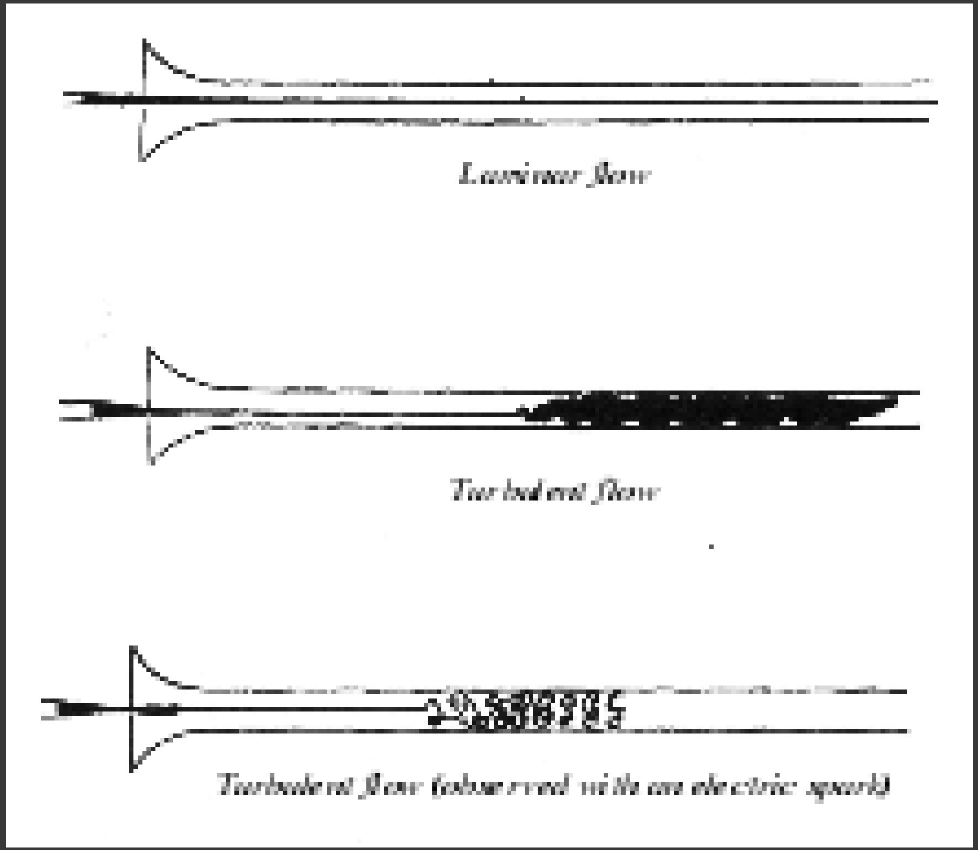
https://www.bronkhorst.com/int/blog-1/what-is-the-difference-between-laminar-flow-and-turbulent-flow/

Ferziger et al. (2020)

Reynolds' Experiment

- In 1883, Osborne Reynolds varied the flow rate of a dyed water jet to study the behavior of water flow.
- The laminar-turbulent transition occurs when 2000 < Re < 13000.

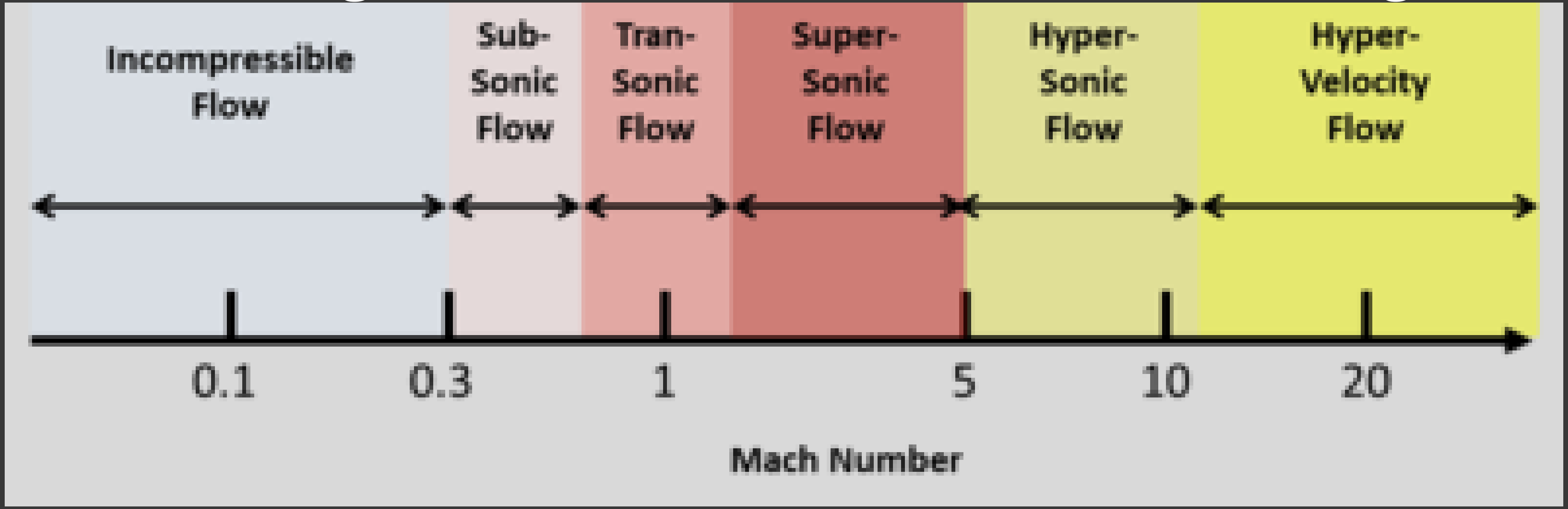




https://en.wikipedia.org/wiki/Laminar%E2%80%93turbulent_transition

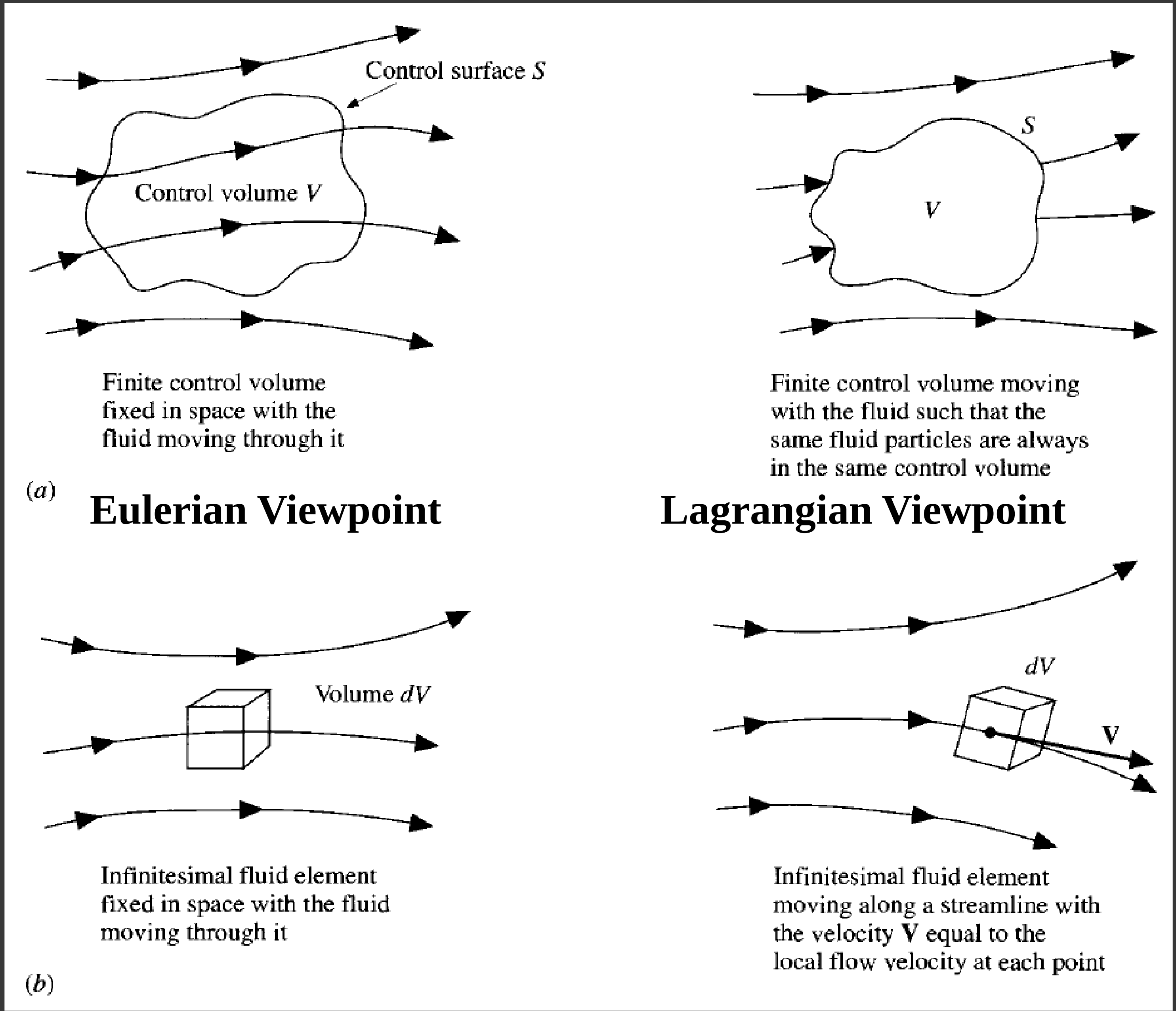
Mach Number Flow Regimes

- The Mach number Ma is the ratio of the local flow speed *u* to the local sound speed *c*: $\mathrm{Ma} = u/c$.
- Incompressible flows (fluid density is considered a constant): $\mathrm{Ma} < 0.3$
- Compressible flows: $\mathrm{Ma} \geq 0.3$
- Hypersonic flows: When $\mathrm{Ma} > 5$, "the compression may create high enough temperatures to change the chemical nature of fluid." Ferziger et al. (2020)



https://en.wikipedia.org/wiki/Compressible_flow#Mach_number,_wave_motion,_and_sonic_speed

Eulerian and Lagrangian Viewpoints



Anderson (1995, p. 41)

Material Derivative

- The material derivative is a total derivative that represents time rate of change due to both local change and convective change.

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \rho}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \rho}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}u + \frac{\partial \rho}{\partial y}v + \frac{\partial \rho}{\partial z}w$$

$$= \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho$$

$$\frac{D}{Dt}\rho = \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)\right]\rho$$

Time Rate of Change

Eulerian viewpoint

$$\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$$

Lagrangian viewpoint

$$\frac{D}{Dt}$$

Local derivative

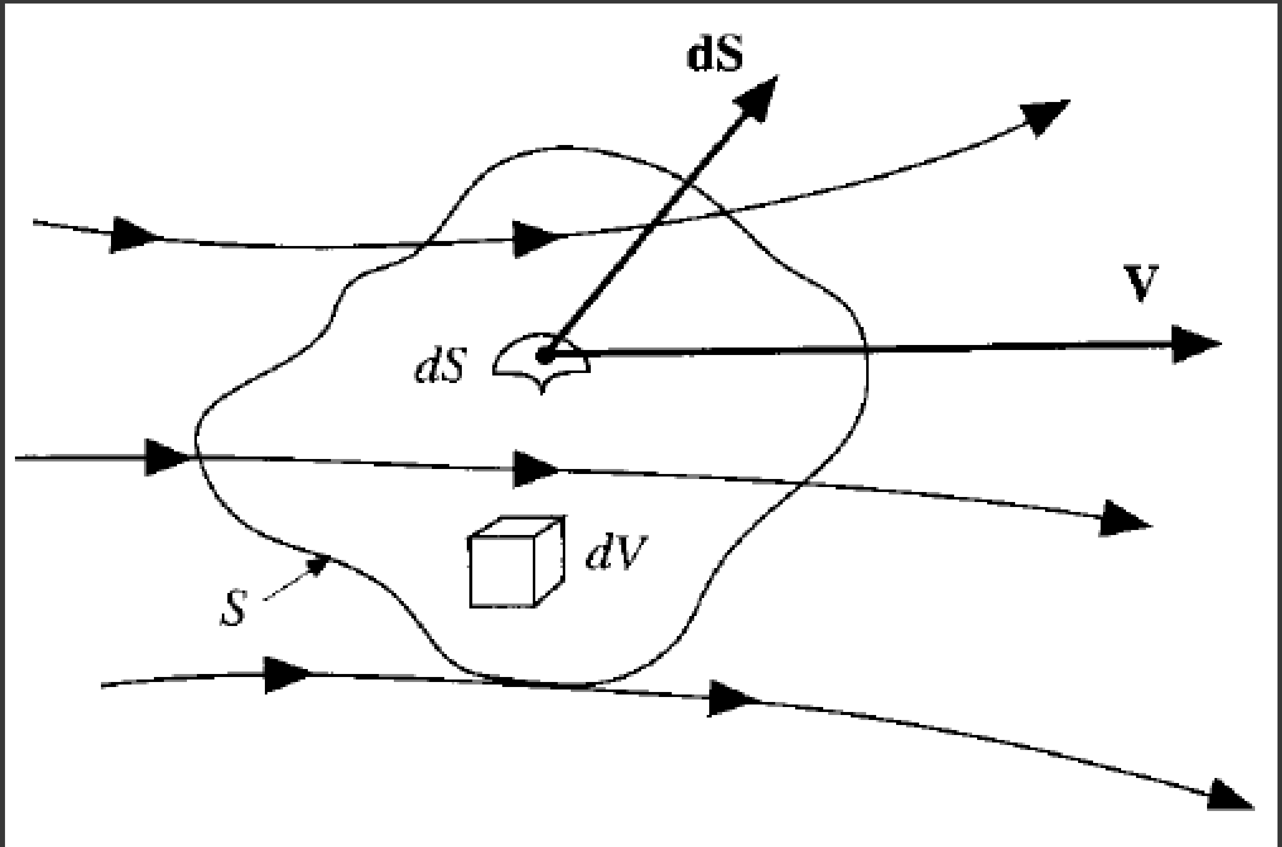
Convective derivative

Scalar quantity: $$\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho = \frac{D\rho}{Dt}$$

Vector quantity: $$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{D\mathbf{v}}{Dt}$$

Continuity Equation: Eulerian Viewpoint

- Consider a finite control volume fixed in space
- Net mass flow out of control volume through surface S is equal to time rate of decrease of mass inside control volume.



Anderson (1995)

Continuity Equation: Eulerian Viewpoint

- Mass flow across a small fixed surface **dS** is $\rho \mathbf{v} \cdot \mathbf{dS}$
- The sign of $\rho \mathbf{v} \cdot \mathbf{dS}$ is positive for an outflow, and is negative for an inflow.

- Net mass flow out of the control volume through surface S is $\iint_S \rho \mathbf{v} \cdot \mathbf{dS}$

- The total mass in the control volume is $\iiint_V \rho dV$

- The time rate of increase of mass is then $\frac{\partial}{\partial t} \iiint_V \rho dV$

- Applying the conservation of mass principle, we obtain the conservative form of the continuity equation $$\frac{\partial}{\partial t} \iiint_V \rho dV + \iint_S \rho \mathbf{v} \cdot \mathbf{dS} = 0$$

Continuity Equation: Eulerian Viewpoint

- The integral form of the continuity equation is

$$\frac{\partial}{\partial t}\iiint_V \rho dV + \iint_S \rho \mathbf{v}\cdot \mathbf{dS} = 0$$

- Applying the divergence theorem to the second term, the surface integral becomes a volume integral and the continuity equation becomes

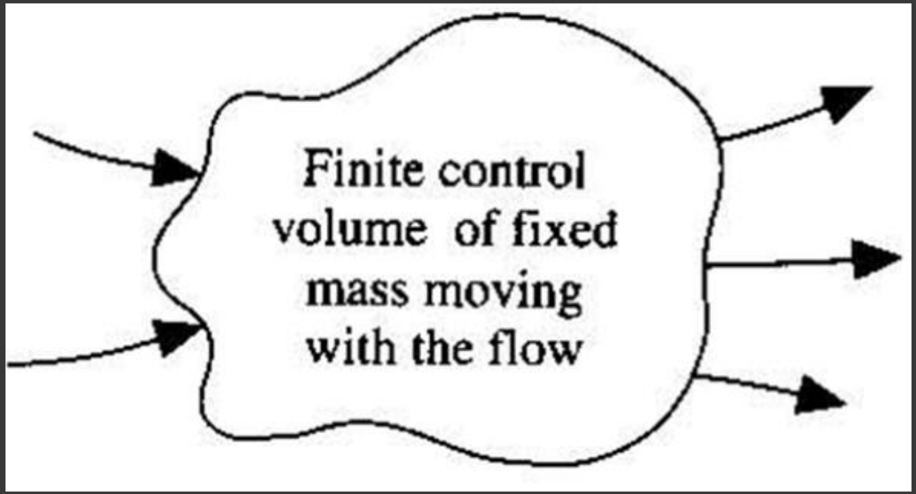
$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right) dV = 0$$

- Since the volume integral is zero, the integrand must vanish.
- So, we obtain the differential form of the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Continuity Equation: Lagrangian Viewpoint

- Let's consider a finite control volume moving with the fluid.
- The total mass of thefinite control volume is $m = \iiint_V \rho dV$
- Since the total mass of finite control volume is always the same, we obtain the nonconservative form of the continuity equation

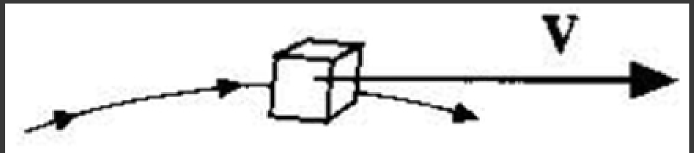


$$\frac{D}{Dt} \iiint_V \rho dV = 0$$

Anderson (1995)

Continuity Equation: Differential Form

Let's consider an infinitesimally small fluid element moving with the flow.
The mass of the fluid element is $\delta m = \rho \delta V$



Since the time rate of change of the mass of fluid element is zero, we obtain which leads to

$$\frac{D(\delta m)}{Dt} = 0$$

$$\frac{D(\delta m)}{Dt} = \frac{D(\rho \delta V)}{Dt} = \delta V \frac{D\rho}{Dt} + \rho \frac{D(\delta V)}{Dt} = 0$$

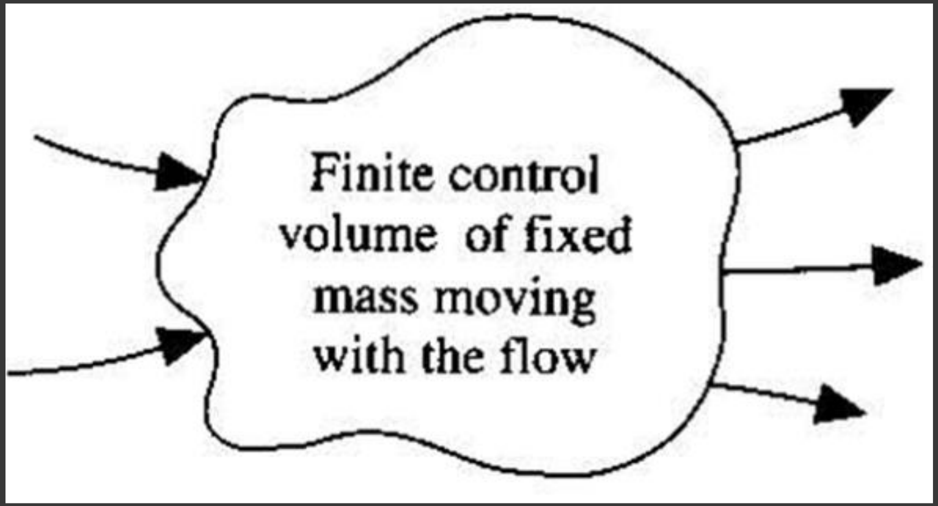
$$\frac{D\rho}{Dt} + \rho \left[\frac{1}{\delta V} \frac{D(\delta V)}{Dt} \right] = 0$$

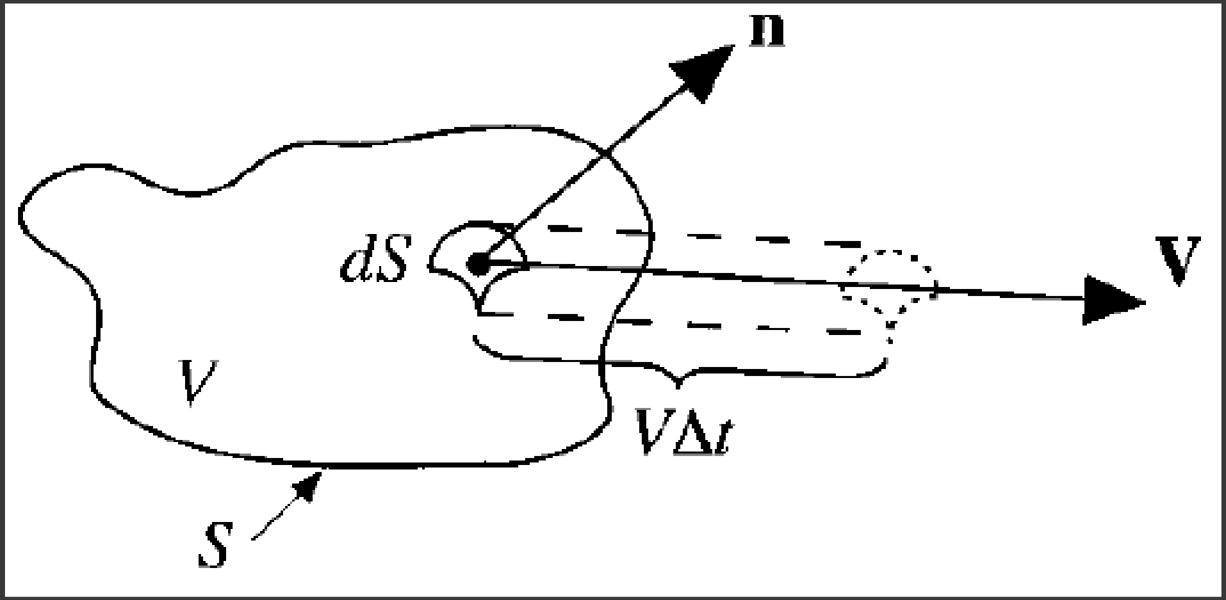
Anderson (1995)

Divergence of Velocity

- Consider a control volume moving with the fluid.
- This control volume has a fixed mass but a changing volume.
- Consider an infinitesimal surface element *dS* moving at local velocity **v**.
- The change in volume due to just the movement of *dS* is

$$\Delta V = \left[\left(\mathbf{v}\Delta t \right) \cdot \mathbf{n} \right] dS = \left(\mathbf{v}\Delta t \right) \cdot \mathbf{dS}$$





Anderson (1995)

Divergence of Velocity

The total change in volume of the whole control volume is the surface integral

$$\Delta V = \iint_S (\mathbf{v}\Delta t)\cdot \mathbf{dS}$$

Dividing by Δt, we obtain the time rate of change of the control volume

$$\frac{DV}{Dt} = \frac{1}{\Delta t}\iint_S (\mathbf{v}\Delta t)\cdot \mathbf{dS} = \iint_S \mathbf{v}\cdot \mathbf{dS} = \iiint_V \nabla\cdot\mathbf{v}\, dV$$

If the moving control volume is shrunk to a very small volume δV becoming an infinitesimal fluid element, we then have

$$\frac{D(\delta V)}{Dt} = \iiint_{\delta V} \nabla\cdot\mathbf{v}\, dV$$

Anderson (1995)

Continuity Equation: Differential Form

Assume that δV is small enough such that $\nabla \cdot \mathbf{v}$ is the same throughout δV. We then obtain

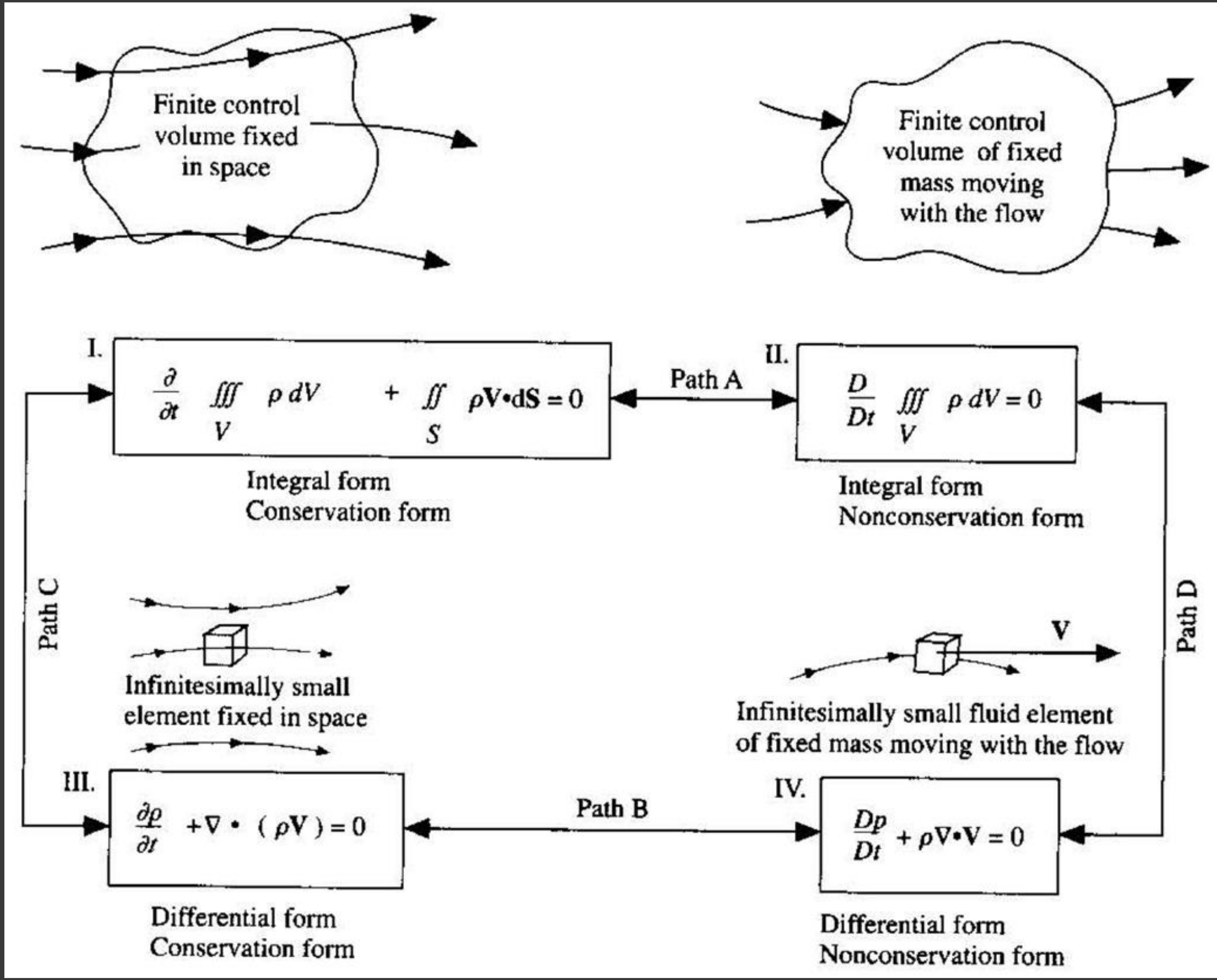
$$\frac{D(\delta V)}{Dt} = (\nabla \cdot \mathbf{v})\,\delta V \rightarrow \nabla \cdot \mathbf{v} = \frac{1}{\delta V}\frac{D(\delta V)}{Dt}$$

Using the previous result, the non-conservative, differential form of continuity equation becomes

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0$$

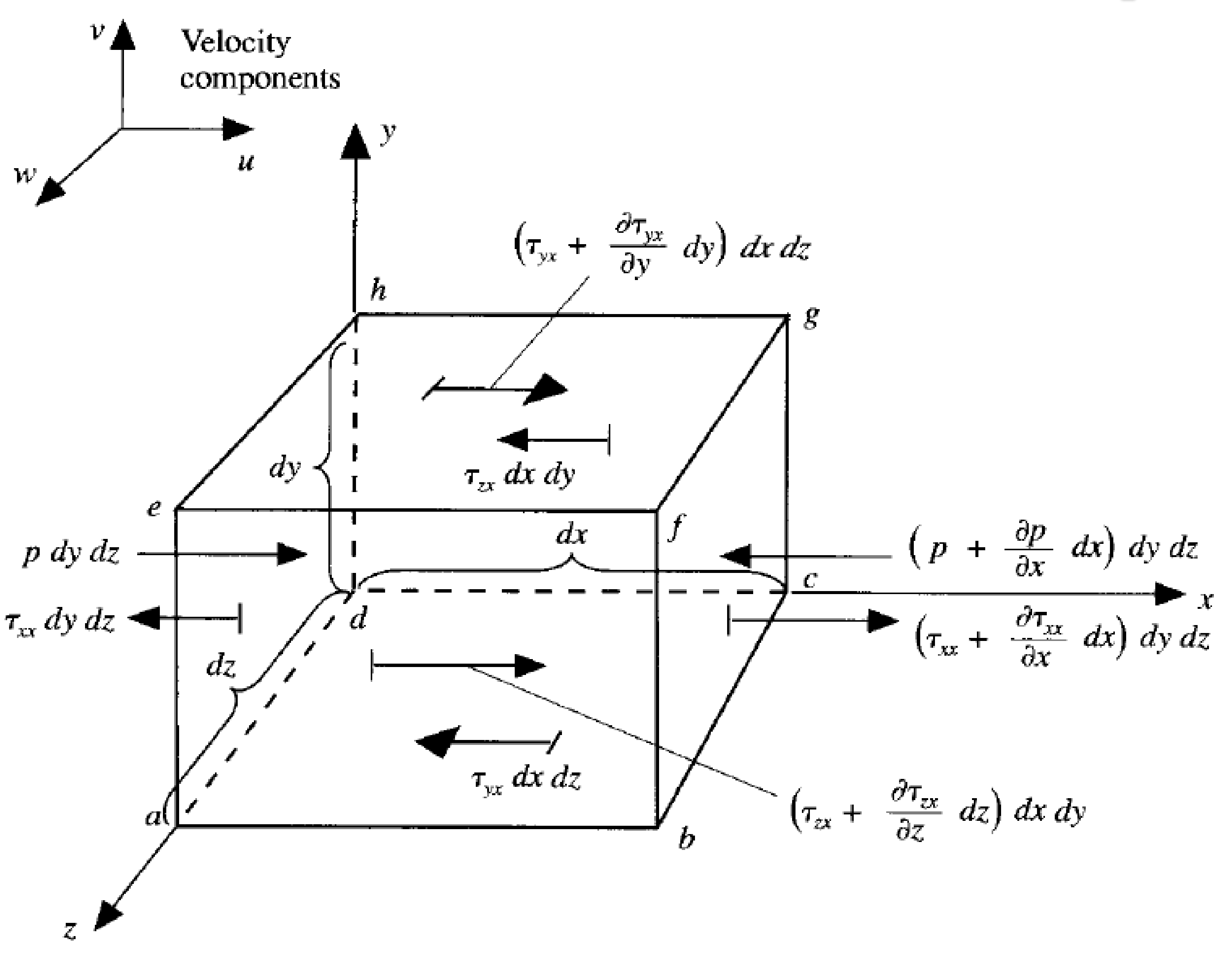
Anderson (1995)

Continuity Equation



Anderson (1995)

Momentum Equation



Consider a fluid element moving with the flow.

Newton's second law

$$\mathbf{F} = m\mathbf{a}$$

Anderson (1995)

Body and Surface Forces

“**Body forces** act directly on the volumetric mass of the fluid element. These forces act at a distance. Examples: gravitational, electric, and magnetic forces.”

“**Surface forces** acting directly on the surface of the fluid element are due to 2 sources:

- Pressure distribution on the surface imposed by the surrounding fluid
- Shear and normal stress distributions imposed outside fluid by means of friction”

Anderson (1995)

Net Body Force

Body force on fluid element acting in

x direction: $\rho f_x (dx\ dy\ dz)$

y direction: $\rho f_y (dx\ dy\ dz)$

z direction: $\rho f_z (dx\ dy\ dz)$

Anderson (1995)

Surface Force in x direction

Net surface force in x direction is

$$\left[p-\left(p+\frac{\partial p}{\partial x}dx\right)\right]dy\ dz+\left[\left(\tau_{xx}+\frac{\partial \tau_{xx}}{\partial x}dx\right)-\tau_{xx}\right]dy\ dz$$

$$+\left[\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}dy\right)-\tau_{xx}\right]dx\ dz+\left[\left(\tau_{zz}+\frac{\partial \tau_{zz}}{\partial z}dz\right)-\tau_{zz}\right]dx\ dy$$

$$=\left[-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right](dx\ dy\ dz)$$

Anderson (1995)

Net Surface Force

Net surface force in

x direction: $$\left[-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right](dx\,dy\,dz)$$

y direction: $$\left[-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right](dx\,dy\,dz)$$

z direction: $$\left[-\frac{\partial p}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right](dx\,dy\,dz)$$

Anderson (1995)

Mass and Acceleration

Mass $\rho\left(dx\ dy\ dz\right)$

Acceleration in

x direction: $a_x = \frac{Du}{Dt}$

y direction: $a_y = \frac{Dv}{Dt}$

z direction: $a_z = \frac{Dw}{Dt}$

Anderson (1995)

Navier-Stokes Equations

Combining the previous results yields momentum equation in nonconservative form

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$

Anderson (1995)

Momentum Conservation

- In the Eulerian viewpoint with a fixed control volume, the momentum conservation gives rise to the equation

$$\frac{\partial}{\partial t}\int_V \rho\mathbf{v}\ \mathrm{d}V + \int_S \rho\mathbf{v}\mathbf{v}\cdot\mathbf{n}\ \mathrm{d}S = \sum\mathbf{F}$$

- The force **F** could be
  - surface forces (pressure, normal and shear stresses, surface tension)
  - body forces (gravity, centrifugal and Coriolis forces, EM forces)
- To make the system of equations close (the number of dependent variables is equal to the number of equations), some assumptions must be made.
- One simple assumption is to assume that the fluid is Newtonian.

Ferziger et al. (2020)

Newtonian Fluids

- The stress tensor T of Newtonian fluids can be written as

$$\mathrm{T} = -\left(p + \frac{2}{3}\mu\nabla\cdot\mathbf{v}\right)I + 2\mu D$$

where p is static pressure, μ is dynamic viscosity, I is unit tensor, D is the rate of strain tensor:

$$\mathrm{D} = \frac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{T}\right]$$

- These two equations can be written in index notation as

$$T_{ij} = -\left(p + \frac{2}{3}\mu\frac{\partial v_j}{\partial x_j}\right)\delta_{ij} + 2\mu D_{ij}$$

$$D_{ij} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right]$$

The Einstein summation convention is used here. The viscous part of the stress tensor is usually denoted as

$$\tau_{ij} = 2\mu D_{ij} - \frac{2}{3}\mu\frac{\partial v_j}{\partial x_j}\delta_{ij}$$

Ferziger et al. (2020)

Momentum Equation

- When there are only stress tensor T and the body force per unit mass b, the momentum conservation equation becomes

$$\frac{\partial}{\partial t}\int_V \rho \mathbf{v} \, \mathrm{d}V + \int_S \rho \mathbf{v}\mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S = \int_S \mathrm{T} \cdot \mathbf{n} \, \mathrm{d}S + \int_V \rho \mathbf{b} \, \mathrm{d}V$$

- Applying the convergence theorem to the surface integrals, we obtain

$$\int_V \left[\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v} - \mathbf{T}) - \rho \mathbf{b} \right] \mathrm{d}V = 0$$

- The integrand must vanish. So, we obtain the differential form of the momentum equation

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \nabla \cdot \mathrm{T} + \rho \mathbf{b}$$

Ferziger et al. (2020)

Conservative Momentum Equation

- The conservative equation for the i^{th} component is

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla\cdot(\rho v_i \mathbf{v}) = \nabla\cdot\mathbf{t}_i + \rho b_i$$

where

$$\mathbf{t}_i = \mu\nabla v_i + \mu(\nabla\mathbf{v})^T\cdot\mathbf{e}_i - \left(p + \frac{2}{3}\mu\nabla\cdot\mathbf{v}\right)\mathbf{e}_i$$

$$= \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\mathbf{e}_j - \left(p + \frac{2}{3}\mu\frac{\partial v_j}{\partial x_j}\right)\mathbf{e}_i$$

and $\mathbf{e}_i$ is the i^{th} Cartesian basis vector.

Ferziger et al. (2020, p. 9)

Nonconservative Momentum Equation

- The nonconservative equation can be obtained as follows.

$$\rho \frac{\partial v_i}{\partial t} + \rho \mathbf{v} \cdot \nabla v_i + v_i \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) \right] = \nabla \cdot \mathbf{t}_i + \rho b_i$$

- Using the differential form of the continuity equation, the bracket term vanishes.
- Then, we obtain the nonconservative momentum equation

$$\rho \frac{\partial v_i}{\partial t} + \rho \mathbf{v} \cdot \nabla v_i = \nabla \cdot \mathbf{t}_i + \rho b_i$$

- This equation is usually solved using the finite difference method.

Ferziger et al. (2020, p. 9)

Conservative Momentum Equation

The conservative equation

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla \cdot (\rho v_i \mathbf{v}) = \nabla \cdot \mathbf{t}_i + \rho b_i$$

can be written in index notation as

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_j v_i)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho b_i$$

If gravity is the only external force, then we have

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_j v_i)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i$$

where g_i is the i^{th} component of gravitational acceleration $\mathbf{g}$.

Ferziger et al. (2020, p. 9)

Navier-Stokes and Euler Equations

- “A viscous flow is one where the transport phenomena of friction, thermal conduction, and/or mass diffusion are included.”
- The continuity, momentum, and energy equations previously mentioned are collectively called the **Navier-Stokes equations**.
- “Inviscid flow is a flow where the dissipative, transport phenomena of viscosity, mass diffusion, and thermal conductivity are neglected, resulting to **Euler equations**."

Ferziger et al. (2020, p. 9)

Dimensionless Form of the Equations

- Using normalization the governing equations can be transformed into a dimensionless form.
- Velocities can be normalized by a reference velocity v_0.
- Spatial coordinates can be normalized by a reference length L.
- Time can be normalized by a reference time t_0.
- Pressure can be normalized by a reference pressure ρv_0^2
- Temperature can be normalized by a temperature difference T_1 -T_0.

$$t^* = \frac{t}{t_0}, x_i^* = \frac{x_i}{l_0}, u_i^* = \frac{u_i}{v_0}, p^* = \frac{p}{\rho v_0^2}, T^* = \frac{T - T_0}{T_1 - T_0}$$

Ferziger et al. (2020, p. 9)

Dimensionless Form of the Equations

"If the fluid properties are constant, the dimensionless form of the continuity, momentum, and temperature equations are"

$$\frac{\partial u_i^*}{\partial x_i^*} = 0, \qquad \mathrm{St}\frac{\partial u_i^*}{\partial t^*} + \frac{\partial \left(u_j^* u_i^*\right)}{\partial x_j^*} = \frac{1}{\mathrm{Re}}\frac{\partial u_i^*}{\partial x_j^{*2}} - \frac{\partial p^*}{\partial x_i^*} + \frac{1}{\mathrm{Fr}^2}\gamma_i$$

$$\mathrm{St}\frac{\partial T^*}{\partial t^*} + \frac{\partial \left(u_j^* T^T\right)}{\partial x_j^*} = \frac{1}{\mathrm{Re}\ \mathrm{Pr}}\frac{\partial^2 T^*}{\partial x_j^{*2}}$$

where the Strouhal number St, the Reynolds number Re, and the Froude number Fr are defined as $\mathrm{St} = \frac{l_0}{v_0 t_0}, \quad \mathrm{Re} = \frac{\rho v_0 l_0}{\mu}, \quad \mathrm{Fr} = \frac{v_0}{\sqrt{l_0 g}}$

and γ_i is the i^{th} component of the normalized gravitational acceleration.

Ferziger et al. (2020, p. 11-12)

Incompressible Flows

- The density of liquids can be considered constant.
- When Ma < 0.3, the density of gases can also be considered constant.
- Flows in such media are said to be incompressible.
- If the flow is isothermal and the viscosity is constant, the continuity and momentum equations reduce to

$$\nabla \cdot \mathbf{v} = 0, \qquad \frac{\partial v_i}{\partial t} + \nabla \cdot (v_i \mathbf{v}) = \nabla \cdot (\nu \nabla v_i) - \frac{1}{\rho} \nabla \cdot (p \mathbf{e}_i) + b_i$$

Ferziger et al. (2020, p. 13)

Inviscid (Euler) Flows

- "In flows far from solid surfaces, the effects of viscosity are usually small."
- "If viscous effects are negligible, i.e., the stress tensor reduces to $\mathrm{T} = -p\mathrm{I}$, the Navier-Stokes equations reduce to the Euler equations."

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \frac{\partial (\rho v_i)}{\partial t} + \nabla \cdot (\rho v_i \mathbf{v}) = -\nabla \cdot (p \mathbf{e}_i) + \rho b_i$$

- In an inviscid flow, the fluid will not stick to walls and slip will occur at solid boundaries.
- "The Euler equations are often used to study compressible flows at high Mach numbers."
- The Euler equations can be solved using a coarser grid than the Navier-Stokes equations due to the absence of boundary layers in which viscosity effects are important.

Ferziger et al. (2020, p. 13-14)

Potential Flows

- In a potential flow, the fluid is assumed to be inviscid and the flow velocity is irrotational, i.e., $\nabla \times \mathbf{v} = 0$
- As a result, there exists a velocity potential ϕ such that $\mathbf{v} = -\nabla \phi$
- In an incompressible flow, the continuity equation becomes the Laplace equation $\nabla \cdot \mathbf{v} = -\nabla \cdot \left(\nabla \phi \right) = -\nabla^2 \phi = 0$
- The velocity vectors are tangential to streamlines which are the lines of constant streamfunction ψ.
- Streamlines are orthogonal to equipotential lines.
- "Potential flows have applications in flows in porous media."
- "The potential theory applied to flow around a body leads to D'Alembert's paradoc, i.e., the body experiences neither drag nor lift."

Ferziger et al. (2020, p. 15)

Creeping (Stokes) Flows

- When Re << 1 (the fluid is very viscous or the object interacting with the fluid is very small), the convection (inertial) terms in the Navier-Stokes equation are very small and can be neglected.
- "The flow is then dominated by the viscous, pressure, and body forces."
- If the fluid properties are constant and the velocities are small, the unsteady terms can be neglected."
- The Navier-Stokes equation becomes the Stokes equations.

$$\nabla \cdot \mathbf{v} = 0, \qquad \nabla \cdot (\mu \nabla u_i) - \frac{1}{\rho} \nabla \cdot (p \mathbf{e}_i) + b_i = 0$$

- "Creeping flows are found in porous media, coating technology, micro-devices, etc."

Ferziger et al. (2020, p. 15)

Boussinesq Approximation

- "In flows accompanied by heat transfer, the fluid properties are normally functions of temperature."
- "If the density variation is not large, the density can be treated as constant in the unsteady and convection terms, and treat it as variable only in the gravitational term."
- "This is called the Boussinesq approximation."
- "The density is usually assumed to vary linearly with temperature."

Ferziger et al. (2020, p. 15)

Incompressible Potential Flows

- Air and water have low viscosity, and in many cases their flows have high Reynolds numbers, i.e., the viscous force is small compared to the inertia force.
- "In flows with high Reynolds numbers, the influence of viscosity is confined to a very thin boundary layer in the immediate neighborhood of the solid wall."
- Thus, the flow outside the boundary layer can be considered inviscid flows.
- Vorticity describing a rotational motion of fluid is defined as $\nabla \times \mathbf{v}$
- "Voriticities are generated by the shearing viscous forces."
- So, flows in the boundary layer are rotational flows.
- Outside the boundary layer, the flow can be considered irrotational, i.e., the vorticity vanishes: $\nabla \times \mathbf{v} = \mathbf{0}$

Biringen and Chow (2011)

Incompressible Potential Flows

- The irrotational condition $\nabla \times \mathbf{v} = \mathbf{0}$ is automatically satisfied if a velocity potential ϕ is defined such that $\mathbf{v} = \nabla \phi$.
- As a result, irrotational flows are also called potential flows.
- If the fluid is incompressible, then $\nabla \cdot \mathbf{v} = 0$
- We then have the Laplace equation $\nabla \cdot \mathbf{v} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0$
- In the problem of an incompressible flow past a body at high Reynolds numbers, the velocity potential outside the boundary layer is computed by solving the Laplace equation with the boundary condition prescribed far upstream and the fluid velocity be tangent to the body surface.
- The fluid velocity within the boundary layer is then obtained by solving the boundary layer equation using the velocity distribution of the external flow along the body surface as the outer boundary condition.

Biringen and Chow (2011)

Incompressible Potential Flows

- "Once the velocity field in the external region is determined, the pressure field p can be computed by solving the Euler equation without body forces"

$$\rho \frac{D\mathbf{v}}{Dt} = \rho\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = -\nabla p$$

- "It is easier to compute the pressure from the Bernoulli equation, the integrated form of the Euler equation."
- For incompressible, irrotational flows, the Bernoulli equation has the form

$$\rho\frac{d\phi}{dt} + p + \frac{1}{2}\rho v^2 = H$$

where the constant of integration H is the Bernoulli constant and $v = |\mathbf{v}|$.

- For steady flows, the Bernoulli equation reduces to $p + \frac{1}{2}\rho v^2 = H$ where H is the stagnation pressure at a point where $v = 0$.

Biringen and Chow (2011)

2D Incompressible Potential Flows

- Inviscid flow is the flow of a fluid with zero viscosity.
- Inviscid flows can be classified as potential flows (irrotational flows) and rotational inviscid flows.
- Potential or irrotational flows has zero vorticity, i.e., $\nabla \times \mathbf{v} = \mathbf{0}$ and the velocity potential ϕ can defined such that $\mathbf{v} = \nabla \phi$
- If a potential flow is also incompressible, i.e., $\nabla \cdot \mathbf{v} = 0$, the potential ϕ satisfies the Laplace equation $\nabla^2 \phi = 0$
- 2D planar flows:
  - Cartesian coordinates: $v_x = \dfrac{\partial \phi}{\partial x}, \quad v_y = \dfrac{\partial \phi}{\partial y}, \quad \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$
  - Polar coordinates: $v_r = \dfrac{\partial \phi}{\partial r}, \quad v_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}, \quad \dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} = 0$

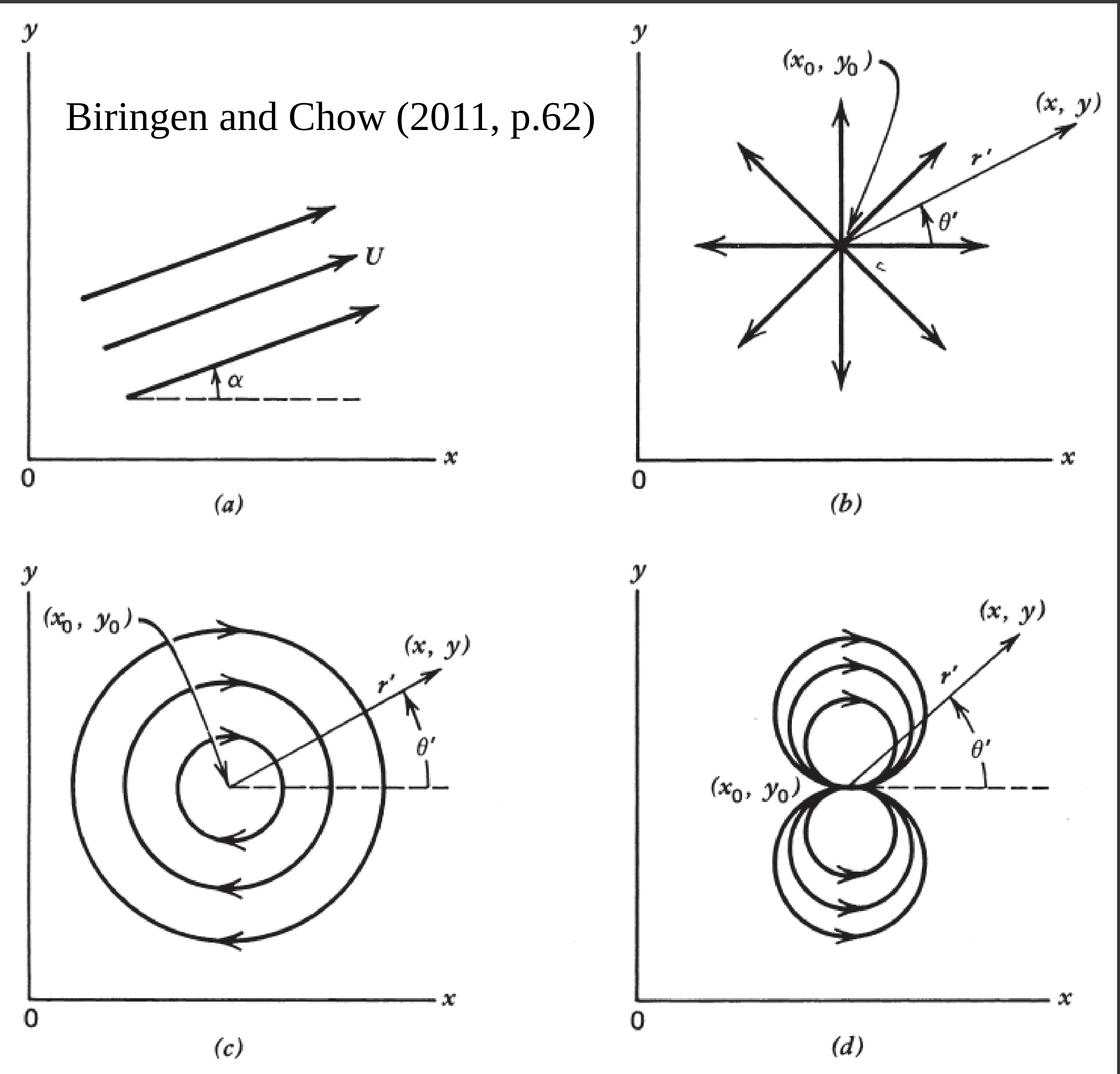
2D Incompressible Potential Flows

- The stream function ψ can also be used instead of the velocity potential.
- The continuity equation $\nabla \cdot \mathbf{v} = 0$ suggests that the velocity can also be expressed interm of the stream function as $\mathbf{v} = \nabla \times \left(\psi \hat{\mathbf{k}}\right)$.
- A line along which ψ = constant is called a streamline.
- Fluid velocities are always tangential to streamlines.
- The irrotational condition becomes $\nabla \times \nabla \times \left(\psi \hat{\mathbf{k}}\right) = \mathbf{0}$
- 2D planar flows:
  - Cartesian coordinates: $v_x = \dfrac{\partial \psi}{\partial y}, \quad v_y = -\dfrac{\partial \psi}{\partial x}, \quad \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} = 0$
  - Polar coordinates: $v_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}, \quad v_\theta = -\dfrac{\partial \psi}{\partial r}, \quad \dfrac{\partial^2 \psi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \psi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \psi}{\partial \theta^2} = 0$

Biringen and Chow (2011, p.53-54)

Elementary Flows

Stream functions corresponding to 4 elementary flows are as follows.



(a) Uniform flow with angle α

$$\psi = U\left(y\cos\alpha - x\sin\alpha\right)$$

(b) Line source with strength Λ

$$\psi = \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) = \frac{\Lambda\theta}{2\pi}$$

(c) Line vortex with circulation Γ:

$$\psi = \frac{\Gamma}{2\pi}\ln\left[\left(x-x_0\right)^2 + \left(y-y_0\right)^2\right]^{\frac{1}{2}} = \frac{\Gamma}{2\pi}\ln r$$

(d) Doublet with strength κ:

$$\psi = -\frac{\kappa}{2\pi}\frac{y-y_0}{\left(x-x_0\right)^2 + \left(y-y_0\right)^2} = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$

Velocity Potentials of Elementary Flows

(a) Uniform flow with angle α:

$$\phi(x,y) = -U\left(x\cos\alpha + y\sin\alpha\right), \qquad \phi(r,\theta) = -Ur\cos\left(\theta - \theta_0\right)$$

(b) Line source with strength Λ:

$$\phi(x,y) = \Lambda\ln\left(x^2 + y^2\right)/4\pi, \qquad \phi(r,\theta) = \Lambda\ln r/2\pi$$

(c) Line vortex with circulation Γ:

$$\phi(x,y) = -\frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right), \qquad \phi(r,\theta) = -\frac{\Gamma\theta}{2\pi}$$

(d) Doublet with strength κ:

$$\phi(x,y) = \frac{\kappa}{2\pi}\frac{x-x_0}{\left(x-x_0\right)^2 + \left(y-y_0\right)^2}, \qquad \phi(r,\theta) = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$$

Velocities of Elementary Flows

(a) Uniform flow with angle α :

$$\mathbf{v}(x,y) = U\cos\alpha\,\hat{\mathbf{i}} + U\sin\alpha\,\hat{\mathbf{j}}$$

(b) Line source with strength Λ

$$\mathbf{v}(x,y) = \frac{\Lambda}{2\pi}\left(\frac{(x-x_0)}{(x-x_0)^2+(y-y_0)^2}\hat{\mathbf{i}} + \frac{(y-y_0)}{(x-x_0)^2+(y-y_0)^2}\hat{\mathbf{j}}\right)$$

(c) Line vortex with circulation Γ:

$$\mathbf{v}(x,y) = \frac{\Gamma}{2\pi}\left(\frac{(y-y_0)}{(x-x_0)^2+(y-y_0)^2}\hat{\mathbf{i}} - \frac{(x-x_0)}{(x-x_0)^2+(y-y_0)^2}\hat{\mathbf{j}}\right)$$

(d) Doublet with strength κ:

$$\mathbf{v}(x,y) = -\frac{\kappa}{2\pi}\left(\frac{(y-y_0)^2-(x-x_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\hat{\mathbf{i}} - \frac{2(x-x_0)(y-y_0)}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\hat{\mathbf{j}}\right)$$

Velocities of Elementary Flows

(a) Uniform flow with angle α :

$$\mathbf{v}(r,\theta) = U\cos\left(\theta-\alpha\right)\hat{\mathbf{r}} - U\sin\left(\theta-\alpha\right)\hat{\boldsymbol{\theta}}$$

(b) Line source with strength Λ

$$\mathbf{v}(r,\theta) = \frac{\Lambda}{2\pi r}\hat{\mathbf{r}}$$

(c) Line vortex with circulation Γ:

$$\mathbf{v}(r,\theta) = -\frac{\Gamma}{2\pi r}\hat{\boldsymbol{\theta}}$$

(d) Doublet with strength κ:

$$\mathbf{v}(r,\theta) = -\frac{\kappa}{2\pi}\frac{\cos\theta}{r^2}\hat{\mathbf{r}} - \frac{\kappa}{2\pi}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\theta}}$$

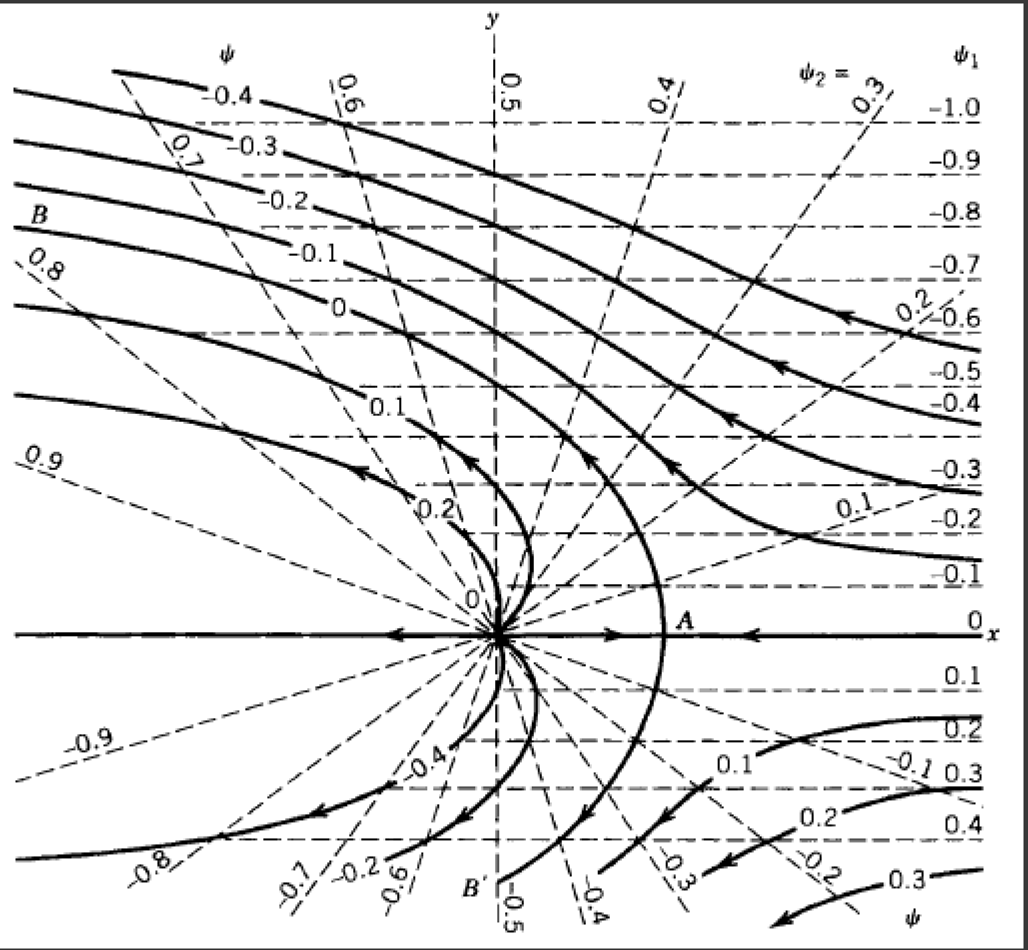
Superposition of Elementary Flows

- If ψ_1 and ψ_2 are solution to $\nabla^2\psi = 0$, $a\psi_1 + b\psi_2$ is also a solution.
- "When a source of strength Λ at $(x_0 - \Delta x, y_0)$ is added to a sink of strength $-\Lambda$ at $(x_0 + \Delta x, y_0)$, a new flow field is obtained."
- "By letting Δx approach zero while keeping the product $2\Delta x\Lambda$ a constant κ, the stream function for a doublet at (x_0, y_0) is obtained."
- "The flow pattern of a double can also be produced by superimposing a vortex at $(x_0, y_0 - \Delta y)$ to a vortex of opposite circulation at $(x_0, y_0 + \Delta y)$, and then letting Δy approach zero."
- The velocities at the center of a line source, a line vortex, and a doublet are infinitely large. The center of these flows are called singularities.
- Singularities do not cause a problem when they are within the boundary of a rigid body.

Biringen and Chow (2011, p.64)

Source in a Uniform Flow

- The stream function for a uniform flow with $\alpha = 0$ is $\psi_1 = -Uy$.
- If a source of strength Λ with ($x_0 = 0$, $y_0 = 0$) and stream function $\psi_2 = \dfrac{\Lambda\theta}{2\pi}$ is superimposed on the uniform flow, the resultant stream function is



$$\psi = Uy + \frac{\Lambda\theta}{2\pi} = U\left(\frac{h\theta}{2\pi} - y\right)$$

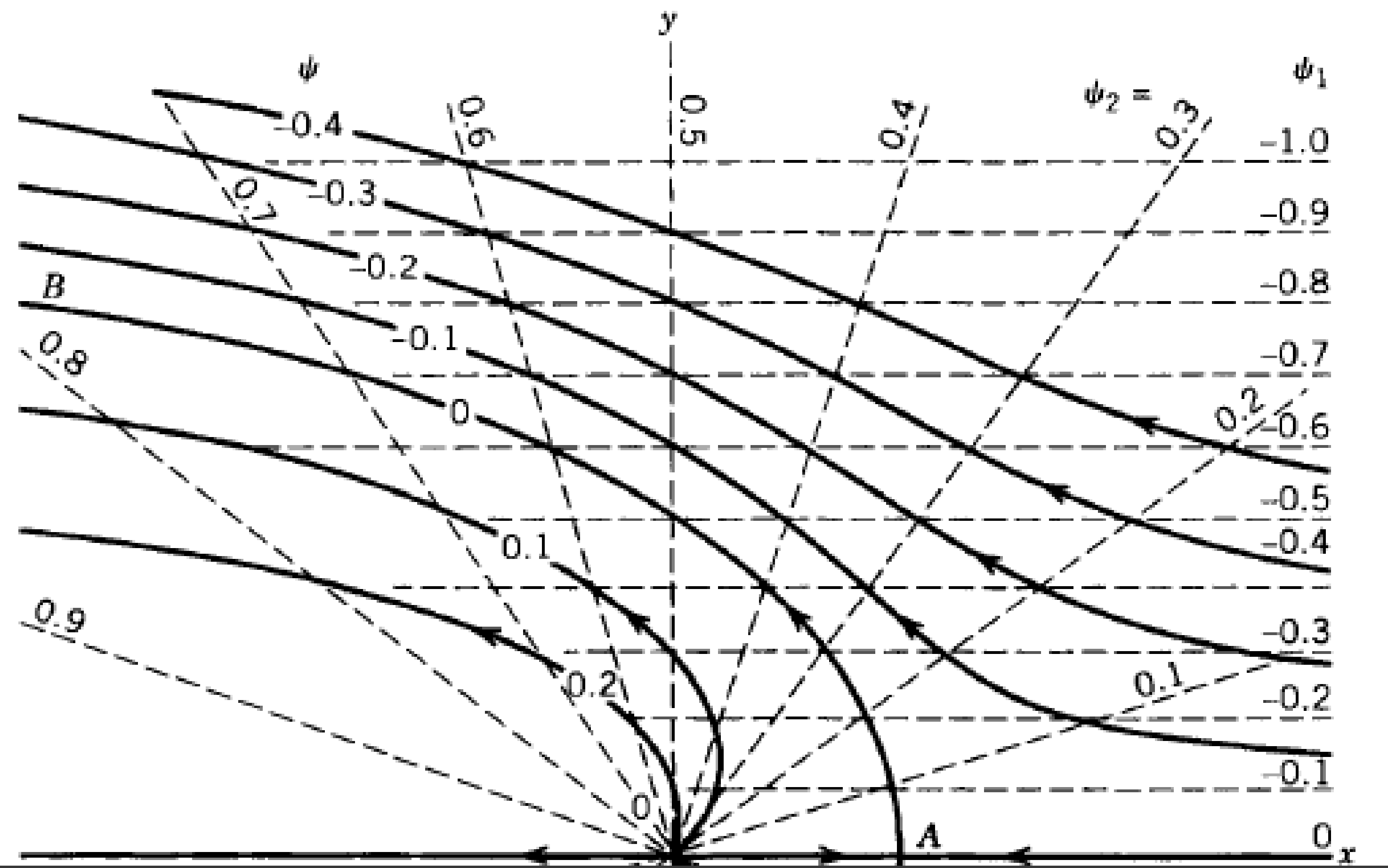
where $h = \Lambda/2U$ is a characteristic length.

- The streamline BAB' with $\psi = 0$ is considered a solid surface enclosing the source.
- "The flow exterior to the surface satisfies the continuity equation and is irrotational."

Kuethe and Chow (1998, p.97)

Source in a Uniform Flow

- "The flow field may be interpreted as that of a horizontal wind past a cliff, whose shape (y_0, θ) is described by the equation $\psi = 0$, that is,



$$y_0 = r_0 \sin\theta = \frac{h\theta}{\pi}, \qquad 0 \le \theta \le \pi$$

where r_0 is the radial distance of a point on the cliff at height y_0 above the x axis."

When $\theta \to -\infty$, $y_0 \to h = \Lambda/2U$.

Kuethe and Chow (1998, p.97)

Source in a Uniform Flow

- With $\theta = \tan^{-1}(y/x)$, the velocity components are

$$v_x = \frac{\partial \psi}{\partial y} = -U + \frac{Uh}{\pi}\frac{x}{x^2 + y^2}$$

$$v_y = -\frac{\partial \psi}{\partial x} = \frac{Uh}{\pi}\frac{y}{x^2 + y^2}$$

- According to these equation, the velocity vanishes ($v_x = v_y = 0$) at point $(h/\pi,0) = (\Lambda/2\pi U,0)$.
- "In other words, the velocity vanishes at point A on the x axis where the velocity from the source, $\Lambda/2\pi x$, cancels the velocity U from the uniform flow."

Kuethe and Chow (1998, p.98)

Flow Pattern of a Source-Sink Pair

- Consider a source of strength Λ at $(-x_0, 0)$ and a sink of strength $-\Lambda$ at $(x_0, 0)$
- The stream function of the combine flow at (x, y) is

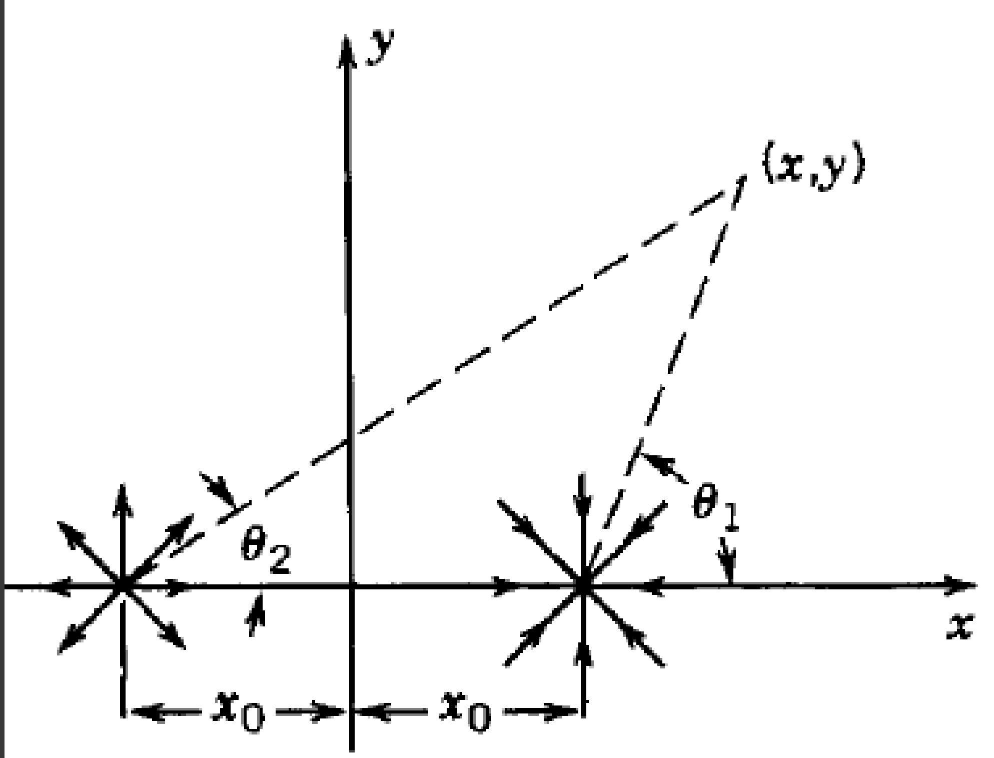
$$\psi = \frac{\Lambda\theta_2}{2\pi} - \frac{\Lambda\theta_1}{2\pi} = \frac{\Lambda}{2\pi}\left(\tan^{-1}\frac{y}{x+x_0} - \tan^{-1}\frac{y}{x-x_0}\right)$$

- Using the trigonometric relation

$$\tan^{-1} A - \tan^{-1} B = \tan^{-1}\frac{A-B}{1+AB}$$

we then obtain

$$\psi = -\frac{\Lambda}{2\pi}\tan^{-1}\frac{2x_0 y}{x^2+y^2-x_0^2}$$

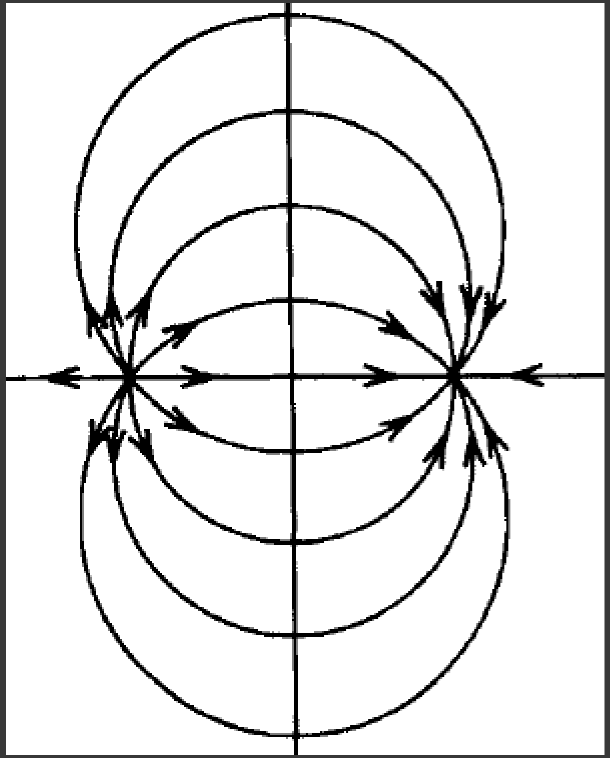


Kuethe and Chow (1998, p.99)

Flow Pattern of a Source-Sink Pair

- Rearranging the equation, we obtain

$$-\tan\frac{2\pi\psi}{\Lambda}=\frac{2x_0 y}{x^2+y^2-x_0^2}$$



$$x^2+y^2+2x_0 y\cot\frac{2\pi\psi}{\Lambda}=x_0^2$$

$$x^2+\left(y+x_0\cot\frac{2\pi\psi}{\Lambda}\right)^2=x_0^2+x_0^2\cot^2\left(\frac{2\pi\psi}{\Lambda}\right)$$

$$=x_0^2\csc^2\left(\frac{2\pi\psi}{\Lambda}\right)$$

- "This equation represents a family of circles with centers on the *y* axis."
- "When $y = 0$, $x = \pm x_0$ for all values of ψ."
- The flow pattern is shown in the figure.

Kuethe and Chow (1998, p.99-100)

Flow Pattern of a Source-Sink Pair

- The flow pattern of a doublet can be obtained "when the distance between the source and sink approaches zero while their strengths approach infinity in such a way that their product remains a constant value of $\kappa = 2x_0\Lambda$."
- As x_0 approaches zero, we have

$$\psi = \lim_{x_0 \to 0}\left[-\frac{\Lambda}{2\pi}\tan^{-1}\frac{2x_0 y}{x^2+y^2-x_0^2}\right] = \lim_{x_0 \to 0}\left[-\frac{1}{2\pi}\frac{(2x_0\Lambda)y}{x^2+y^2-x_0^2}\right]$$

$$= -\frac{\kappa}{2\pi}\frac{y}{x^2+y^2} = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$

- In the limit, the velocity potential is $\phi = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$

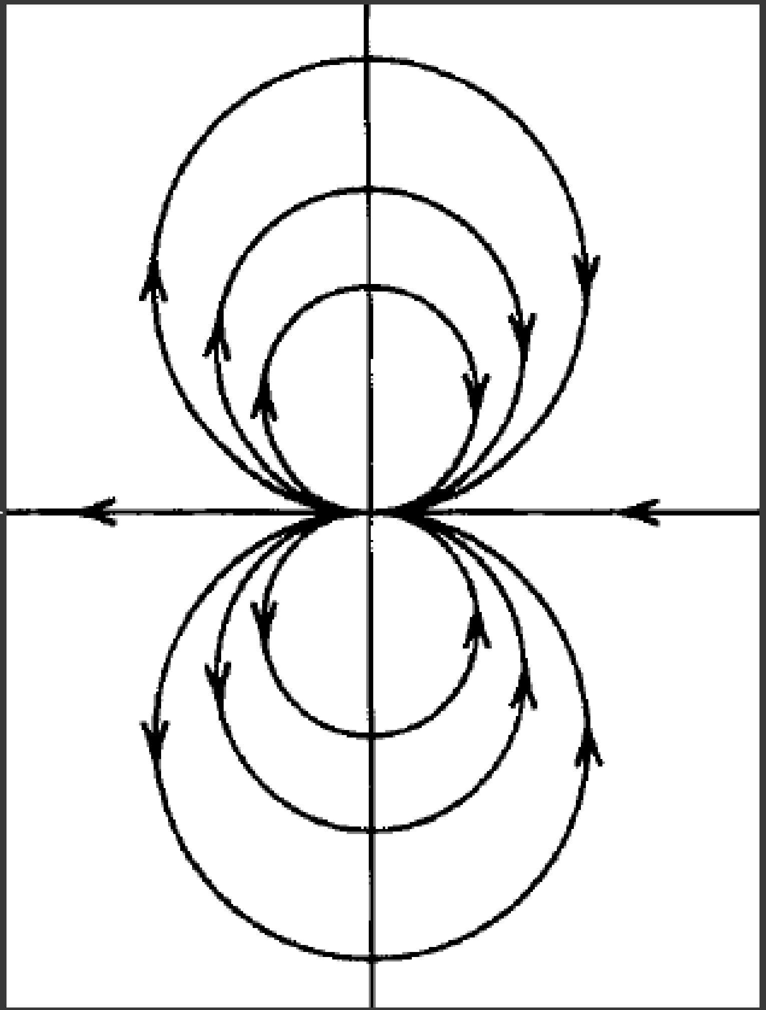
Kuethe and Chow (1998, p.100)

Flow Pattern of a Source-Sink Pair

- The streamlines of a doublet flow of strength κ (lines of constant ψ) are circles as can be seen by rearranging the equation in the form

$$x^2 + \left(y + \frac{\kappa}{4\pi\psi} \right)^2 = \left(\frac{\kappa}{4\pi\psi} \right)^2$$

- Each circle has a center are at $(0, -\kappa/4\pi\psi)$ and a radius of $\kappa/4\pi\psi$.
- All streamlines (circles) pass through the origin.



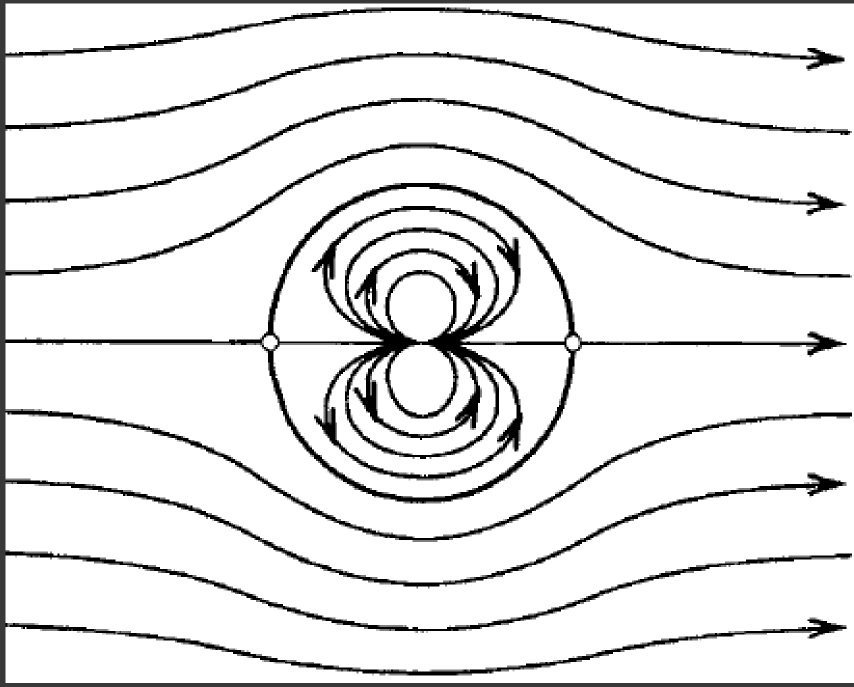
Kuethe and Chow (1998, p.101)

Flow Past a Circular Cylinder

- "The stream function for a uniform flow with velocity U in the direction of the positive x axis is $\psi = Uy$."
- "If the uniform flow is added to a doublet, the flow about a circular cylinder in a uniform stream is obtained."
- The resulting stream function is $\psi = Uy - \dfrac{\kappa y}{2\pi r^2} = Uy\left(1 - \dfrac{\kappa}{2\pi U r^2}\right)$
- Let $\kappa/2\pi U = a^2$. Then, we have

$$\psi = Uy\left(1 - \frac{a^2}{r^2}\right)$$

- "The zero streamline consists of the x axis and a circle of radius $r = a$."



Kuethe and Chow (1998, p.101)

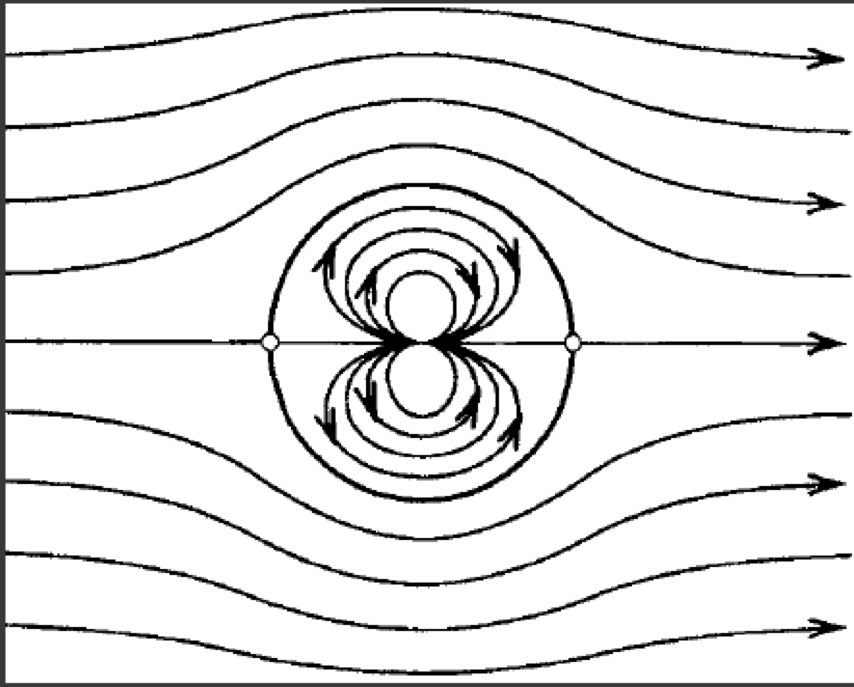
Flow Past a Circular Cylinder

- With $y = r\sin\theta$, the velocity components are

$$v_r = U\left(1-\frac{a^2}{r^2}\right)\cos\theta, \qquad v_\theta = -U\left(1+\frac{a^2}{r^2}\right)\sin\theta$$

- On the cylinder surface ($r = a$), $u_r = 0$ and $u_\theta = -2U\sin\theta$.
- The pressure distribution on the surface is given by the Bernoulli equation

$$C_p = \frac{p-p_\infty}{q_\infty} = 1-\left(\frac{u_\theta}{U}\right)^2 = 1-4\sin^2\theta \quad \text{for } r = a$$



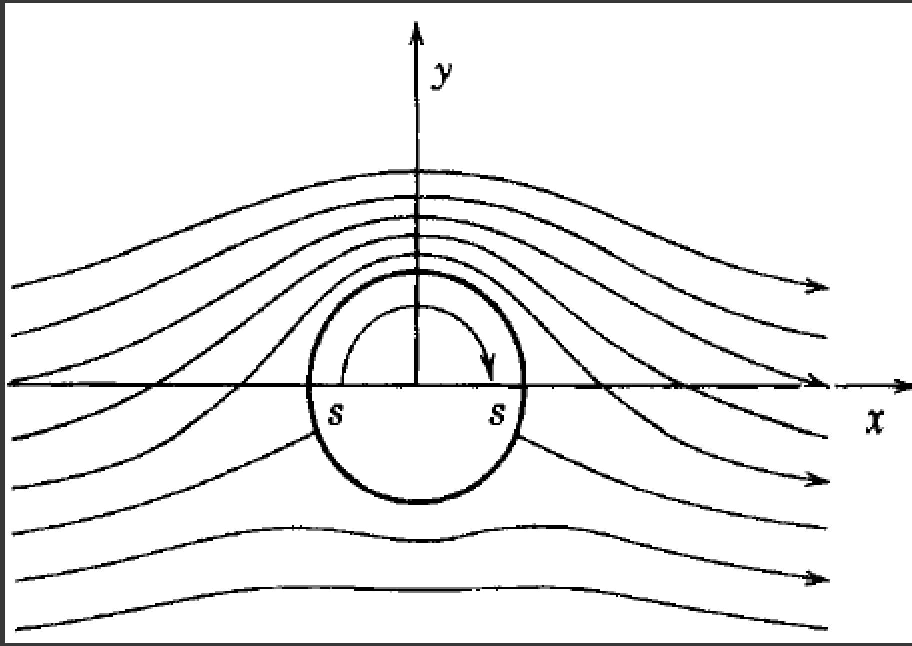
where C_p is called the pressure coefficient.

Kuethe and Chow (1998, p.101-102)

Circulatory Flow about a Cylinder

- "If a stream function for a vortex at origin is added to $\psi = Uy\left(1 - a^2/r^2\right)$, the resulting stream function will satisfy the continuity, irrotationality, the boundary conditions for the circulatory flow about a circular cylinder in a uniform stream:"

$$\psi = Uy\left(1 - \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right)$$



- "The uniform stream is in the direction of the positive *x* axis, and the circulatory flow is clockwise."
- The zero streamline corresponds to a cylinder of radius $r = a$.

Kuethe and Chow (1998, p.102)

Circulatory Flow about a Cylinder

- The velocity components for this flow is as follows.

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -\frac{\partial \psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}$$

- On the cylinder surface ($r = a$), we have

$$u_r = 0, \qquad u_\theta = -2U\sin\theta - \frac{\Gamma}{2\pi a}$$

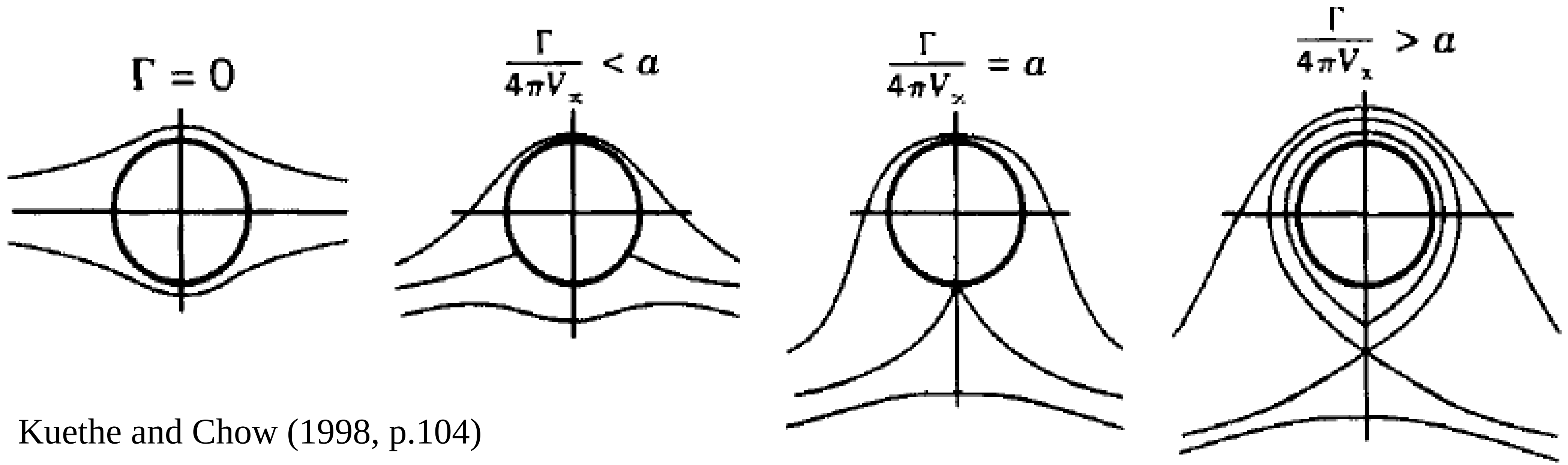
- u_θ vanishes when $\theta = \theta_s$: $\quad \sin\theta_s = -\frac{\Gamma}{4\pi a U}$
- Since $\sin\theta = y/r$, the stagnation points are

$$x_s = \pm\sqrt{a^2 - y_s^2}, \qquad y_s = -\frac{\Gamma}{4\pi U}$$

Kuethe and Chow (1998, p.102-103)

Circulatory Flow about a Cylinder

- "As Γ becomes large, the stagnation points move downward until $(\Gamma/4\pi U)^2$ equals a^2; for this condition, the stagnation points coincide on the y axis at $(0, -a)$."
- When $(\Gamma/4\pi U)^2 > a^2$, the stagnation points leave the body and the equations $x_s = \pm\sqrt{a^2 - y_s^2}$, $y_s = -\dfrac{\Gamma}{4\pi U}$ no longer hold.



Kuethe and Chow (1998, p.104)

Exercise

- Use a linear combination of the 4 elementary flows of your choice to generate a flow pattern.
- Plot streamlines and equipotential lines for the flow.
- Also plot the velocity fields.

Steady Potential Flows in 2D

- The velocity components can be computed from the velocity potential ϕ and stream function ψ as follows.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

- Both ϕ and ψ satisfy the Laplace equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

- Curves $\phi =$ constant are called equipotential lines.
- Curves $\psi =$ constant are called streamlines.
- In 2D flows, the equipotential lines and streamlines form two families of mutually orthogonal curves since

$$\begin{aligned} \nabla \phi \cdot \nabla \psi &= \phi_x \psi_x + \phi_y \psi_y \\ &= (u)(-v) + (u)(v) = 0 \end{aligned}$$

Biringen and Chow (2011, p.76)

Analytic Function

- Let's construct a complex function from the velocity potential and the stream function as

$$w(z) = \phi(x, y) + i\psi(x, y), \qquad z = x + iy, \quad i = \sqrt{-1}$$

- Since the real and imaginary parts of *w* satisfy the Cauchy-Riemann conditions, i.e.,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x},$$

  w(*z*) is complex-differentiable through out the region occupied by the fluid.
- In other words, *w*(*z*) is an analytic function of *z*.
- "If we choose an arbitrary analytic function *w*(*z*), the real and imaginary parts of the function are then qualify as the velocity potential and stream function of a potential flow in the *x*-*y* plane."

Biringen and Chow (2011, p.76-77)

Complex Potential

- The complex function $w(z)$ is called the complex potential, whose derivative is related to the velocity components as $dw/dz = u - iv$
- So, the magnitude of the velocity vector is $|\mathbf{v}| = |dw/dz| = \sqrt{u^2 + v^2}$
- The complex potential corresponding to the uniform flow is

$$w(z) = Uze^{-i\alpha} = U(x\cos\alpha + y\sin\alpha) + iU(y\cos\alpha - x\sin\alpha)$$

- The complex potential of the source is $w(z) = (\Lambda/2\pi)\log(z - z_0)$
- The complex potential of the vortex is $w(z) = (i\Gamma/2\pi)\log(z - z_0)$
- The complex potential of the doublet is $w(z) = \kappa/2\pi(z - z_0)$
- Here, $z_0 = x_0 + iy_0$.
- The complex potential of the source, vortex, and doublet has a singular point at z_0, where the first derivative of the function is unbounded.

Biringen and Chow (2011, p.77)

Exercise

- Using the complex potentials given in the previous slide, plot the equipotential lines and streamlines of the 4 elementary flows and a linear combination of elementary flows.

Conformal Mapping

- The principle of superposition of elementary flows can be applied to the complex potential to generate new flows.
- For example, the sum of Uz (uniform horizontal flow) and $\kappa/2\pi z$ (doublet at origin) represents the complex potential of a uniform flow past a circular cylinder of radius $\sqrt{\kappa/2\pi U}$
- The method of conformal mapping can generate new flow patterns using coordinate transformations.
- Let $z = f(z')$ where f is an analytic function of z'.
- We then have

$$\frac{dw}{dz'} = \frac{dw}{dz}\frac{dz}{dz'} = \frac{dw}{dz}\frac{df}{dz'}$$

Biringen and Chow (2011, p.78)

Conformal Mapping

- "After being transformed into the z' plane, the complex potential can be written in terms of the new coordinates as"

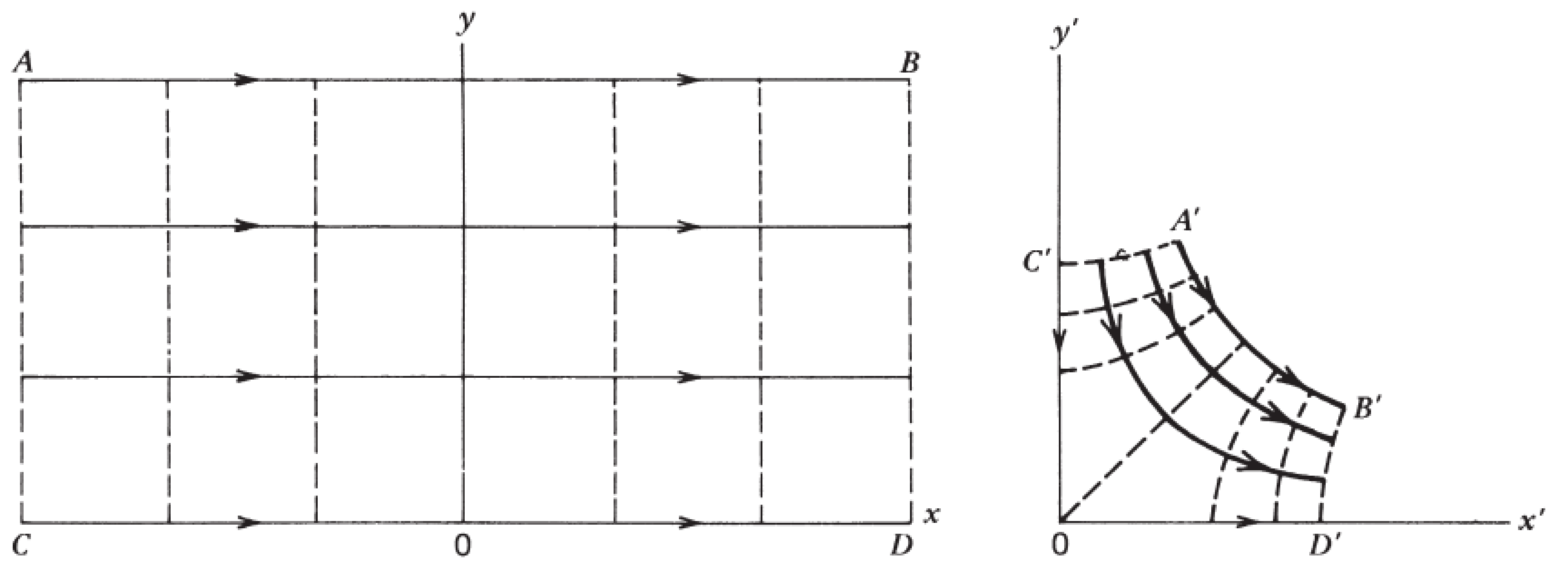
$$w\left[f(z')\right] = \phi'\left(x', y'\right) + i\psi'\left(x', y'\right)$$

- The curves ϕ' = constant and ψ' = constant remain mutually orthogonal in the x'-y' plane after the transformation or mapping.
- Thus, this is called conformal mapping.

Biringen and Chow (2011, p.78)

Conformal Mapping: Example

- Consider the complex potential $w(z) = Uz = Ux + iUy$ of a uniform flow with speed U in the positive x direction. Let the mapping be $z = z'^2$.
- The complex potential becomes $w(z') = Uz'^2 = U\left(x'^2 - y'^2\right) + i2Ux'y'$
- "The equipotential lines $x = c$ (dashed lines) and streamlines $y = k$ (solid lines) in the x-y plane are mapped into equipotential lines $x'^2 - y'^2 = c$ and streamlines $2x'y' = k$ in the x'-y' plane, respectively."



Biringen and Chow (2011, p.80)

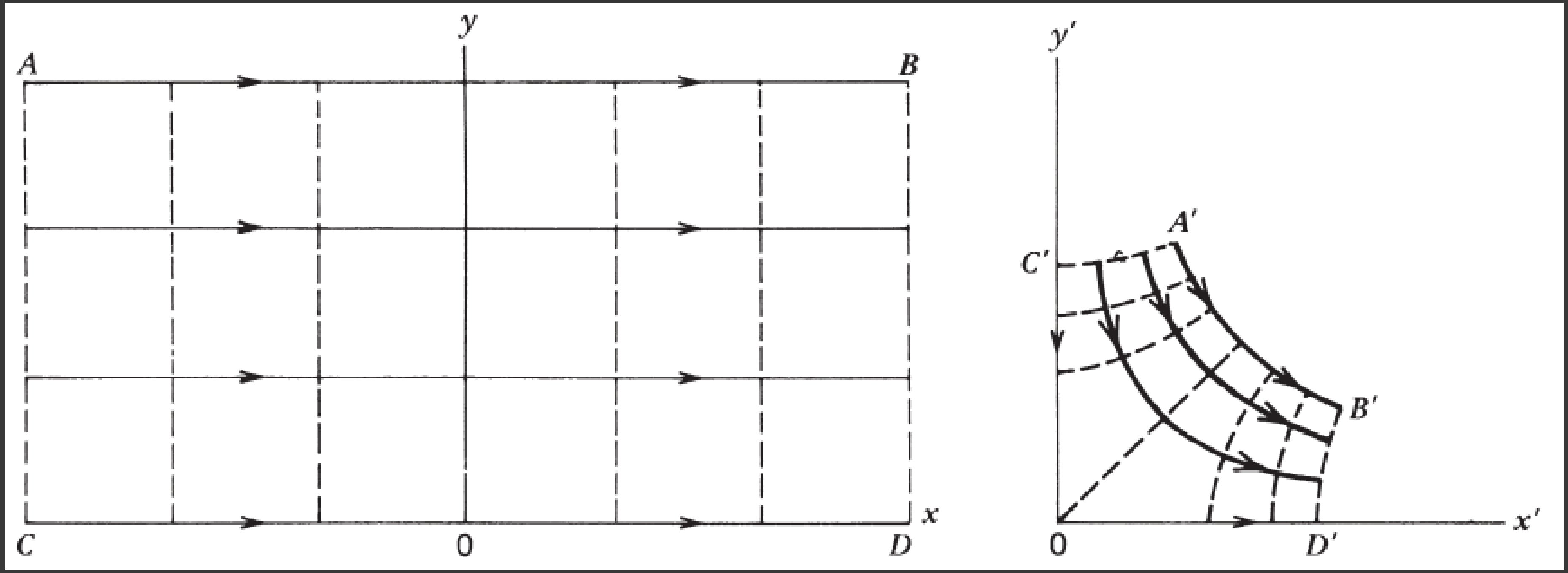
Exercise

Plot the equipotential lines and streamlines of the uniform flow

$$w(z) = Uz = Ux + iUy$$

in the original domain, and those of the corresponding flow in the transformed domain

$$w(z') = Uz'^2 = U\left(x'^2 - y'^2\right) + i2Ux'y'$$



Biringen and Chow (2011, p.80)

Exercise

Plot the equipotential lines and streamlines of the source flow, vortex flow, and doublet flow after using the conformal mapping $z = z'^2$.

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