Methods for Solving 2-point Boundary Value Problems

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August 20, 2020
Boundary Value Problems

- To obtain a unique solution to a differential equation, conditions on the solution or its derivative must be specified.
- If the conditions are specified at a single point, we have an initial value problem.
- If the conditions are specified at more than one point, we have a boundary value problem (BVP).
- For ODE, conditions are specified at 2 points leading to a two-point BVP.

Heath (2002, p. 422)
First-order 2-point BVP

- “Since a higher-order ODE can always be transformed to a first-order system of ODEs, so it suffices to consider only the first-order case.”
- A general first-order 2-point BVP for an ODE has the form

\[ y' = f(x, y), \quad a < x < b \]

with boundary conditions

\[ g(y(a), y(b)) = 0 \]

Heath (2002, p. 423)
First-order 2-point BVP

“Boundary conditions are separated if any component of \( g \) involves solution values only at \( a \) or at \( b \).”

“Boundary conditions are linear if they have the form

\[
B_a y(a) + B_b y(b) = c
\]

Example: Separated linear boundary conditions

2\(^{nd}\)-order scalar BVP \( y'' = f(x, y, y') \), \( y(a) = \alpha, y(b) = \beta \)

is equivalent to 1\(^{st}\)-order system

\[
\begin{bmatrix}
y_1' \\
y_2'
\end{bmatrix} = \begin{bmatrix} y_2 \\ f(x, y_1, y_2) \end{bmatrix}
\]

with separated linear boundary conditions

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix} y_1(a) \\
y_2(a)
\end{bmatrix} + \begin{bmatrix} 0 & 0 \\
1 & 0
\end{bmatrix} \begin{bmatrix} y_1(b) \\
y_2(b)
\end{bmatrix} = \begin{bmatrix} \alpha \\
\beta
\end{bmatrix}
\]

Heath (2002, p. 423)
First-order 2-point BVP

"For the general first-order 2-point BVP
\[ y' = f(x, y), \quad a < x < b \]
with boundary conditions
\[ g(y(a), y(b)) = 0, \]
let \( y(x, y_a) \) denote the solution to the ODE with initial condition \( y(a) = y_a \) for \( y_a \in \mathbb{R}^n \)."

“For a given \( y_a \), the solution \( y(x, y_a) \) of the IVP is a solution of the BVP if the system of nonlinear algebraic equations
\[ h(y_a) \overset{\text{def}}{=} g(y_a, y(b, y_a)) = 0, \]
has a unique solution.”

Heath (2002, p. 424)
The shooting method replaces a given BVP by a sequence of IVPs.

A first-order two-point BVP is equivalent to the system of nonlinear algebraic equations

$$h(y_a) \overset{\text{def}}{=} g(y_a, y(b, y_a)) = 0,$$

“One way to solve the BVP is to solve the nonlinear system $h(y_a) = 0$.”

“Evaluation of $h(y_a)$ requires solving an IVP to determine $y(b, y_a)$.”

Consider the BVP \( y'' = f(x, y, y'), y(a) = \alpha, y(b) = \beta \)

The initial slope \( y'(a) \) is varied until the solution to the IVP at \( x = \beta \) matches the desired boundary value. The boundary conditions are

\[
g(y(a), y(b)) = \begin{bmatrix} y_1(a) - \alpha \\ y_1(b) - \beta \end{bmatrix} = 0
\]

The nonlinear system to be solved is

\[
h(y_a) = \begin{bmatrix} y_1(a; y_a) - \alpha \\ y_1(b; y_a) - \beta \end{bmatrix} = 0
\]

Heath (2002, p. 428)
“The first component of $h(y_a)$ will be zero if $x_1 = \alpha$.”
“So, we must solve the scalar nonlinear equation in $x_2$,
\[ h_2(\alpha, x_2) = y_1(b; \alpha, x_2) = 0 \]
for which we can use a root finding algorithm.”

Heath (2002, p. 428)
Example

Consider the two-point BVP for the 2\textsuperscript{nd}-order ODE
\[ y'' = 6x, \quad 0 < x < 1 \]
with boundary conditions
\[ y(0) = 0, \quad y(1) = 1 \]
which can be transformed into a system of 1\textsuperscript{st}-order ODEs
\[
\begin{bmatrix}
  y_1' \\
  y_2'
\end{bmatrix} =
\begin{bmatrix}
  y_2 \\
  6x
\end{bmatrix}
\]
where \( y_1(x) = y(x) \) and \( y_2(x) = y'(x) \)
The next step is to guess the initial slope value \( y_2(0) \) and to solve the corresponding IVP for \( y_1(1) \). Then vary the initial slope until the right boundary condition is satisfied.

Heath (2002, p. 429)
Example: We want $y_1(1) = 1$

First trial: using RK4 with $h = 0.5$ and $y_2(0) = 1$

\[
y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\]
\[
y_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{6} \left( \begin{bmatrix} 0.5 \\ 0.0 \end{bmatrix} + 2 \begin{bmatrix} 0.50 \\ 0.75 \end{bmatrix} + 2 \begin{bmatrix} 0.6875 \\ 0.7500 \end{bmatrix} + \begin{bmatrix} 0.875 \\ 1.500 \end{bmatrix} \right) = \begin{bmatrix} 0.625 \\ 1.750 \end{bmatrix}.
\]

Second trial: using RK4 with $h = 0.5$ and $y_2(0) = -1$

\[
y_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{1}{6} \left( \begin{bmatrix} -0.5 \\ 0.0 \end{bmatrix} + 2 \begin{bmatrix} -0.50 \\ 0.75 \end{bmatrix} + 2 \begin{bmatrix} -0.3125 \\ 0.7500 \end{bmatrix} + \begin{bmatrix} -0.125 \\ 1.500 \end{bmatrix} \right) = \begin{bmatrix} -0.375 \\ -0.250 \end{bmatrix}.
\]

Heath (2002, p. 429)
Example: We want $y_1(1) = 1$

**Third trial:** using RK4 with $h = 0.5$ and $y_2(0) = 0$

\[
y_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{6} \left( \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix} + 2 \begin{bmatrix} 0.00 \\ 0.75 \end{bmatrix} + 2 \begin{bmatrix} 0.1875 \\ 0.7500 \end{bmatrix} + \begin{bmatrix} 0.375 \\ 1.500 \end{bmatrix} \right) = \begin{bmatrix} 0.125 \\ 0.750 \end{bmatrix}
\]

\[
y_2 = \begin{bmatrix} 0.125 \\ 0.750 \end{bmatrix} + \frac{1}{6} \left( \begin{bmatrix} 0.375 \\ 1.500 \end{bmatrix} + 2 \begin{bmatrix} 0.75 \\ 2.25 \end{bmatrix} + 2 \begin{bmatrix} 0.9375 \\ 2.2500 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3.0 \end{bmatrix} \right) = \begin{bmatrix} 1.0 \\ 3.0 \end{bmatrix}
\]

Heath (2002, p. 429-430)
 Finite Difference Method

- “The shooting method solves a BVP by approximately satisfying the ODE from the beginning and iterates until the boundary conditions are satisfied.”
- “The finite difference (FD) method satisfies the boundary conditions from the beginning and iterates until the ODE is approximately satisfied.”
- “The finite difference converts a BVP into a system of algebraic equations rather than a sequence of IVPs as in the shooting method.”
- “In a FD method, a set of mesh points within the domain is introduced and then any derivatives appearing in the ODE or boundary conditions are replaced by FD approximations at the mesh points.”

\[ y_2(0) = 0 \]

Heath (2002, p. 431)
Example: 2 Dirichlet BCs

Consider the BVP

\[ y'' = f(x, y, y'), \quad a < x < b \]

with Dirichlet boundary conditions

\[ y(a) = \alpha, \quad y(b) = \beta \]

We introduce mesh points \( x_i = a + ih, \quad i = 0, \ldots, n + 1 \)

where \( h = (b - a)/(n + 1) \) and seek approximate solution values \( y_i \approx y(x_i), \quad i = 1, \ldots, n \)

The derivatives in the ODE are replaced by 2\textsuperscript{nd}-order FD approximations

\[
y'(x_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}, \quad y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}
\]

Heath (2002, p. 431)
The ODE then becomes a system of algebraic equations

\[
\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f \left( x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h} \right), \quad i = 1, \ldots, n
\]

“The system of algebraic equations resulting from a FD method for a two-point BVP may be linear or nonlinear, depending on whether \( f \) is linear or nonlinear in \( y \) and \( y' \).”
Consider the BVP
\[ y'' = f(x, y, y'), \quad a < x < b \]
with boundary conditions
\[ y(a) = \alpha, \quad y'(b) = \beta \]
Here, the mesh points are \( x_i = a + ih, \quad i = 0, \ldots, n \)
where \( h = (b - a)/n \) and seek approximate solution values \( y_i \approx y(x_i), \quad i = 1, \ldots, n \)
The derivatives in the ODE are replaced by 2\textsuperscript{nd}-order FD approximations
\[ y'(x_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}, \quad y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \]
Adapted from an example given in Heath (2002, p. 431)
Example: Dirichlet-Neumann BCs

The ODE then becomes a system of algebraic equations

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f \left( x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h} \right), \quad i = 1, \ldots, n-1$$

The right BC contributes the last algebraic equation

$$y'(b) \approx \frac{-0.5y_{n-2} + 2y_{n-1} - 1.5y_n}{h} = \beta$$

to the system of equations. Here, a one-sided 2\textsuperscript{nd}-order FD approximation is used.

Adapted from an example given in Heath (2002, p. 431)
Consider the two-point BVP
\[ y'' = 6x, \quad 0 < x < 1 \]
with boundary conditions
\[ y(0) = 0, \quad y(1) = 1 \]
Let the mesh points are \( x_0 = 0, \ x_1 = 0.5, \ x_2 = 1 \).
From the boundary conditions, we know that
\[ y_0 = y(0) = 0, \quad y_2 = y(1) = 1. \]
We then only seek an approximate solution \( y_1 \approx y(0.5) \).
Using finite difference approximations at \( x = 0.5 \), the ODE becomes
\[
\frac{y_2 - 2y_1 + y_0}{h^2} = f \left( x_1, y_1, \frac{y_2 - y_0}{2h} \right)
\]
Heath (2002, p. 431-432)
Example: FD

“Substituting the boundary data, mesh size, and right-hand side function for this example, we obtain”

\[
\frac{1 - 2y_1 + 0}{(0.5)^2} = 6x_1
\]

Rearranging this yields \( y(0.5) \approx y_1 = 0.125 \)

Heath (2002, p. 431-432)
Collocation Method

“For a scalar two-point BVP

\[ y'' = f(x, y, y'), \quad a < x < b \]

with boundary conditions

\[ y(a) = \alpha, \quad y'(b) = \beta \]

we seek an approximate solution of the form

\[ y(x) \approx v(x, c) = \sum_{i=1}^{n} c_i \phi_i(x) \]

where the \( \phi_i \) are basis functions defined on \([a, b]\) and \( c \) is an \( n \)-vector of parameters to be determined.”

“Popular choices of basis functions include polynomials, B-splines, and trigonometric functions.”

“To determine the vector of parameter $c$, we define a set of $n$ points $a = x_1 < \cdots < x_n = b$, called collocation points, and force the approximate solution to satisfy the ODE at the interior collocation points and the boundary conditions at the end points.”

“The simplest choice of collocation points is to use an equally-spaced mesh.” This choice is suitable if the basis functions are trigonometric functions.

“If the basis function are polynomials, then the Chebyshev points will provide greater accuracy.”

Heath (2002, p. 433)
“Having chosen collocation points and smooth basis functions that we can differentiate analytically, we can now substitute the approximate solution and its derivatives into the ODE at each interior collocation points to obtain a set of algebraic equations

\[ v''(x_i, c) = f(x_i, v(x_i, c), v'(x_i, c)), \quad i = 2, \ldots, n - 1 \]

while enforcing the boundary conditions yields two additional equations

\[ v(x_1, c) = \alpha, \quad v(x_n, c) = \beta \]

“The system of \( n \) equations in \( n \) unknowns is then solved for the parameter vector \( c \) that determines the approximate solution function \( v \).”
Consider the two-point BVP

\[ y'' = 6x, \quad 0 < x < 1 \]

with boundary conditions

\[ y(0) = 0, \quad y(1) = 1 \]

Let the collocation points are \( x_0 = 0, x_1 = 0.5, x_2 = 1 \).

Using the first three monomials as the basis functions, the approximate solution has the form

\[ v(x, c) = c_1 + c_2 x + c_3 x^2 \]

“The derivatives of this function are given by”

\[ v'(x, c) = c_2 + 2c_3 x, \quad v''(x, c) = 2c_3 \]

Heath (2002, p. 434)
Example: Collocation Method

Requiring the ODE to be satisfied at the interior collocation point \( x_2 = 0.5 \) gives the equation

\[ \nu''(x_2, c) = f(x_2, \nu(x_2, c), \nu'(x_2, c)) \]

or

\[ 2c_3 = 6x_2 = 6(0.5) = 3 \]

Requiring the left BC to be satisfied at \( x_1 = 0 \) gives

\[ c_1 + c_2 x_1 + c_3 x_1^2 = c_1 = 0 \]

Requiring the right BC to be satisfied at \( x_3 = 1 \) gives

\[ c_1 + c_2 x_3 + c_3 x_3^2 = c_1 + c_2 + c_3 = 1 \]

Solving this linear system yields

\[ c_1 = 0, \quad c_2 = -0.5, \quad c_3 = 1.5 \]

The approximate solution is

\[ \nu(x, c) = -0.5x + 1.5x^2 \]

Heath (2002, p. 431-432)
Example: Collocation Method

For this problem, the true solution is $y(x) = x^3$

The figure below shows the true solution (solid line) and the collocation solution (dashed line).

Heath (2002, p. 435)
Collocation Method

- “Satisfying the differential equation at a given point is not the same as agreeing with the exact solution to the differential equation at that point, since two functions can have the same slope at a point without having the same value there.”

- “Thus, we do not expect the approximate solution to be exact at the collocation points.”

- When the basis functions have global support (basis functions are nonzero over the entire domain), this yields a spectral method.

- When the basis functions have compact support, this yields a finite element method.

Heath (2002, p. 435)
Weighted Residual Method

- Collocation solutions satisfy differential equations at collocation points -- the residual is zero at these points.
- We can minimize the residual over the entire interval of integration.
- “Consider the scalar Poisson equation in one dimension

\[ y'' = f(x), \quad a < x < b \]

with homogeneous boundary conditions”

\[ y(a) = y(b) = 0 \]

We also seek an approximate solution of the form

\[ y(x) \approx v(x, c) = \sum_{i=1}^{n} c_i \phi_i(x) \]

Heath (2002, p. 436)
Weighted Residual Method

Substituting the approximate solution into the differential equation yields the residual

\[ r(x, c) = v''(x, c) - f(x) = \sum_{i=1}^{n} c_i \phi_i''(x) - f(x) \]

“The weighted residual method forces the residual to be orthogonal to each of a given set of weight functions \( w_i \),

\[ \int_{a}^{b} r(x, c) w_i(x) dx = 0, \quad i = 1, \ldots, n \]

which yields a linear system \( Ax = b \) whose solution given the vector of parameters \( c \).”

Heath (2002, p. 437)
The collocation method is a weighted residual method in which the weight functions are the Dirac delta functions

\[ w_i(x) = \delta(x - x_i) \]

That is

\[ \int_a^b r(x, c) \delta(x - x_i) \, dx = r(x_i, c) = 0, \quad i = 1, \ldots, n \]

Heath (2002, p. 437)
The least squares method minimize the function

\[ F(c) = \frac{1}{2} \int_a^b r(x, c)^2 \, dx \]

by setting each component of its gradient to zero

\[
0 = \frac{\partial F}{\partial c_i} = \int_a^b r(x, c) \frac{\partial r}{\partial c_i} \, dx = \int_a^b r(x, c) \phi'_i(x) \, dx
\]

\[
= \int_a^b \left( \sum_{j=1}^n c_j \phi_j(x) - f(x) \right) \phi'_i(x) \, dx
\]

\[
= \sum_{j=1}^n \left( \int_a^b \phi_j(x) \phi'_i(x) \, dx \right) c_j - \int_a^b f(x) \phi'_i(x) \, dx
\]

Heath (2002, p. 437)
Least Squares Method

which is a symmetric system of linear algebraic equations

\[ Ax = b \]

where

\[ A_{ij} = \int_a^b \phi_i(x)\phi_j''(x)\,dx, \quad b_i = \int_a^b f(x)\phi_i''(x)\,dx \]

“These integrals can be evaluated either analytically or by numerical integration.”

The least squares method is a weighted residual method in which the weight functions are

\[ w_i(x) = \frac{\partial r}{\partial c_i} \]
In the Galerkin method, the weight functions are chosen to be the same as the basis functions, that is, \( w_i = \phi_i \).

“With this choice of weight functions, the orthogonality condition becomes

\[
\int_a^b r(x, c)\phi_i(x)dx = 0, \quad i = 1, \ldots, n
\]

or

\[
\int_a^b v''(x, c)\phi_i(x)dx = \int_a^b f(x)\phi_i(x)dx, \quad i = 1, \ldots, n
\]

Using integration by parts yields

\[
\int_a^b v''(x, c)\phi_i(x)dx = v'(x, c)\phi_i(x)|_a^b - \int_a^b v'(x, c)\phi_i'(x)dx
\]

\[
= v'(b, c)\phi_i(b) - v'(a, c)\phi_i(a) - \int_a^b v'(x, c)\phi_i'(x)dx
\]

Heath (2002, p. 438)
The orthogonality condition becomes

\[ v'(b)\phi_i(b) - v'(a)\phi_i(a) - \int_a^b v'(x)\phi_i'(x)\,dx = \int_a^b f(x)\phi_i(x)\,dx \]

Assuming the basis function satisfy the homogeneous boundary conditions \( \phi_i(0) = \phi_i(1) = 0 \), the orthogonality condition then becomes

\[ -\int_a^b v'(x)\phi_i'(x)\,dx = \int_a^b f(x)\phi_i(x)\,dx \]

\[ -\int_a^b \left( \sum_{j=1}^n c_j \phi_j'(x) \right) \phi_i'(x)\,dx = \int_a^b f(x)\phi_i(x)\,dx \]

\[ -\sum_{j=1}^n \left( \int_a^b \phi_j'(x)\phi_i'(x)\,dx \right) c_j = \int_a^b f(x)\phi_i(x)\,dx \]

Heath (2002, p. 438)
which is a symmetric system of linear algebraic equations

\[ \mathbf{Ax} = \mathbf{b} \]

where

\[ A_{ij} = \int_{a}^{b} \phi'_i(x)\phi'_j(x) \, dx, \quad b_i = \int_{a}^{b} f(x)\phi_i(x) \, dx \]

In structural analysis, \( \mathbf{A} \) is called the stiffness matrix and \( \mathbf{b} \) is called the load vector.

“Like collocation, the Galerkin method can be used with basis function having global support (i.e., a spectral method) or local support (i.e., a finite element method).”

Heath (2002, p. 438)
Consider the two-point BVP
\[ y'' = 6x, \quad 0 < x < 1 \]
with boundary conditions
\[ y(0) = 0, \quad y(1) = 1 \]
Let use piecewise linear polynomials as basis functions with knots located at \( x_0 = 0, x_1 = 0.5, x_2 = 1 \).

Heath (2002, p. 439)
Example: Galerkin Method

“We seek an approximate solution of the form”

\[ y(x) \approx v(x, c) = c_1 \phi_1(x) + c_2 \phi_2(x) + c_3 \phi_3(x) \]

“From the boundary conditions, we must have”

\[ c_1 = 0, \quad c_2 = 1. \]

“To obtain the remaining parameter \( c_2 \), we impose the Galerkin orthogonality condition on the interior basis function \( \phi_2 \) and obtain the equation”

\[
- \sum_{j=1}^{3} \left( \int_{0}^{1} \phi_j'(x) \phi_2'(x) \, dx \right) c_j = \int_{0}^{1} 6x \phi_2(x) \, dx \\
2c_1 - 4c_2 + 2c_3 = 3/2 \rightarrow c_2 = 1/8
\]

The approximate solution is \( v(x, c) = 0.125 \phi_2(x) + \phi_3(x) \)

Example: Galerkin Method

This figure shows the true solution (solid line) and the piecewise linear Galerkin solution (dashed line).